Monday, August 24, 2020

Class philosophy: High expectations, high support.

Norms: camera on. I will call on you. Lots of collaboration.
always open to negotiation within reason.

Grading scheme:

~~Exams~~	Quizzes (every 2 weeks)	65%
~~Quizzes~~	Check-ins (Tues + Fri)	10%
WebAssign		10%
Written HW		10%
Projects		5%

Drop policy is very generous. Details on syllabus on Piazza.

Technology

- OneNote. Your CUid comes with access
- Piazza. Main source of class communication
  Anonymous posting!
  Ask questions any time and answer peers' questions.
  Go answer question about tablet as "attendance"
- Mathematica Install tonight

WRITING MATHEMATICS WELL

Communicating mathematics well is an important part of doing mathematics. As you write up your homework solutions, keep these things in mind:

- Write in sentences.
  Complete thoughts are sentences that end in periods. You may still highlight important equations by displaying them, but even displayed equations should have punctuation! Use paragraphs to separate important ideas.
- Use helpful connective phrases.
  "If", "then", "so", "therefore", "we see that", "recall that", ...
- Your audience is other students in the class who have not seen this problem before.
  Remind the reader of any relevant facts from class or the book. Your solution should give adequate detail so that the reader can follow your solution.
- It is possible to write too much!
  If you write out every triviality, the reader may get lost in the details. This is not good writing, either. (In particular, really trivial calculations need not be shown.)
- Avoid shorthand.
  Don't use arrows, and write out 'for all', 'there exists'.
- You may wish to outline your problem-solving strategy at the beginning of the problem.

Example. Here are two different solutions to the same problem. Which one is easier to understand?

$$(0-3)^2 + (x-2)^2 = 25$$
$$3^2 = 9 + (x^2 - 4x + 4) = 25$$
$$x^2 - 4x - 12$$
$$(x-6)(x+2) \implies x = -2, 6 \quad x > 0 \quad x = 6$$

WHY THIS IS POORLY WRITTEN:

- You don't know what problem the writer was solving.
- You can't tell what's an assumption and what's a conclusion.
- Where does one thought end and another begin? There are no sentences!
- In the 2nd line: combining two thoughts can create untruths (3^2 is 9 but it isn't 25).
- The 3rd line dangles; what's being asserted here? It's not a sentence.
- What's the relationship between all these phrases? Connective phrases would help!

Problem. Find a point in the plane on the positive x-axis that has distance 5 from the point $(2, 3)$.

Solution. $\boxed{\text{The desired point is } (6, 0).}$

To find this, we note if $(x, 0)$ is a solution, then x must must satisfy the equation $(x-2)^2 + (0-3)^2 = 25$, which follows from the planar distance formula between the points $(x, 0)$ and $(2, 3)$. It follows that $x^2 - 4x + 13 = 25$. Then

$$x^2 - 4x - 12 = 0.$$

Factoring, we obtain

$$(x-6)(x+2) = 0,$$

satisfied by either $x = -2$ or $x = 6$. Since we assumed $x > 0$ and $y = 0$, we see $(6, 0)$ is the desired point.

WHY THIS IS WELL-WRITTEN:

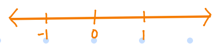
- The writer described the problem, and strategy for solution.
- Every thought is a complete sentence with subject and verb (the "equals" sign is a verb).
- She answered the question right at the beginning. (Boxing answers is customary.)
- Notice even the equations have punctuation (comma, periods) as they are part of sentences.
- She highlighted important ingredients, displayed important equations, avoided trivial algebra.

Writing well will benefit you, too! It helps you structure your own thinking, and you will thank yourself when you re-read your solutions later.

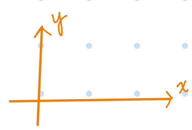
9.1 Three dimensional coordinate systems.

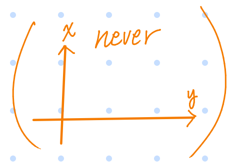
$\mathbb{R}$ is the symbol for real numbers, i.e. numbers without i.

$\mathbb{R}^1$ is the real line

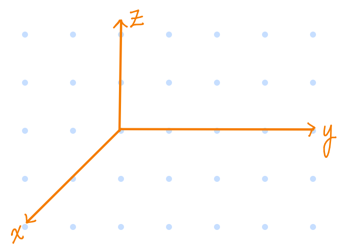


$\mathbb{R}^2$ is the xy-plane

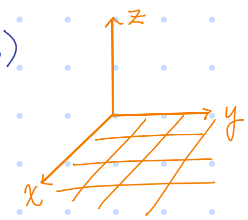




$\mathbb{R}^3$ is xyz-space, or 3-space
(Note: always use right-hand rule in this class)

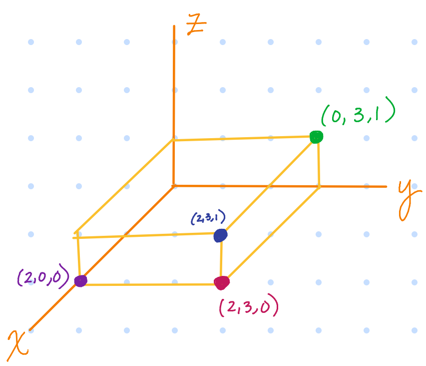


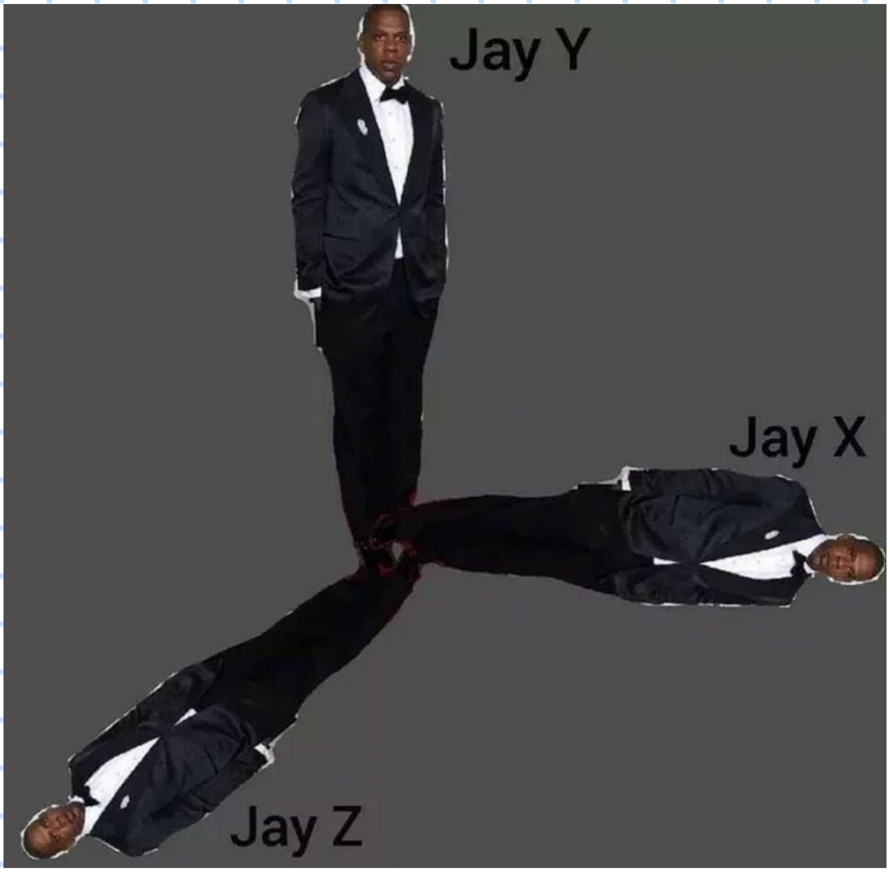
ex.) Draw xy-plane (same as $z=0$)



Pay attn to how to draw grid!

ex) Plot $(2, 3, 1)$ and its projections on the xy-plane, the yz-plane, and the x-axis





Right-hand rule

or

Left-hand rule?

Wednesday, August 25

9.2 Vectors

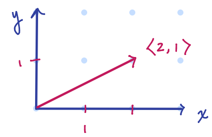
A <u>vector</u> is a mathematical object with magnitude and direction

(Note: The zero vector has magnitude 0 and no direction)

Notation: $\vec{u}$, $\vec{v}$ or $\mathbf{u}$, $\mathbf{v}$

$\vec{u} = \langle 7, -7, 4 \rangle$ components or $7\vec{i} - 7\vec{j} + 4\vec{k}$

Graph:



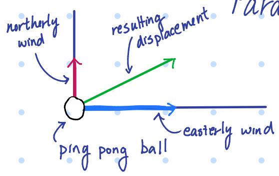
Meaning: velocity, force, displacement, and more!

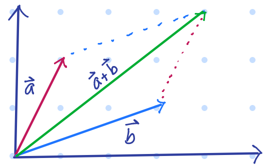
Algebra with vectors

	Intuition	Geometrically	Algebraically

Adding vectors

"Parallelogram Rule"

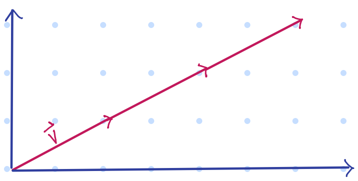


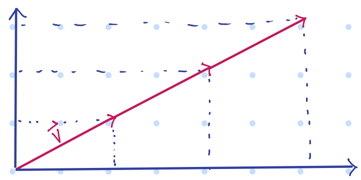


$\vec{a} = \langle 1, 2 \rangle$, $\vec{b} = \langle 3, 1 \rangle$

$\vec{a} + \vec{b} = \langle 4, 3 \rangle$

Scalar multiplication

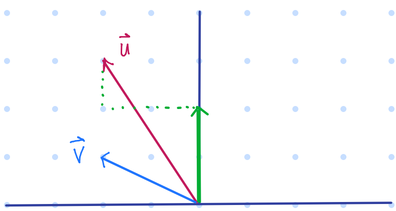




$\vec{v} = \langle 2, 1 \rangle$

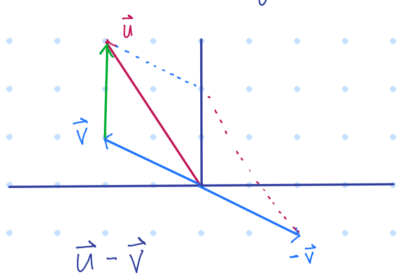
$3\vec{v} = \langle 6, 3 \rangle$

Subtracting vectors



$\vec{u} - \vec{v}$

"Triangle Rule"



$\vec{u} - \vec{v}$

$\vec{u} = \langle -2, 3 \rangle$, $\vec{v} = \langle -2, 1 \rangle$

$\vec{u} - \vec{v} = \langle 0, 2 \rangle$

The <u>magnitude</u> or <u>norm</u> of a vector $\vec{v} = \langle a, b, c \rangle$ is denoted $|\vec{v}|$

The magnitude is computed via $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$

A unit vector is a vector with magnitude 1

To scale up/down any vector $\vec{v}$ into a unit vector with the direction as $\vec{v}$, take $\vec{v}$ and divide by $|\vec{v}|$.

Friday, August 27

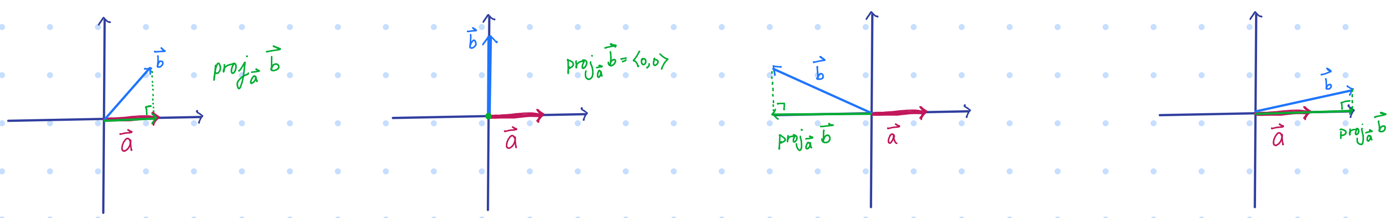
Reminders

- WebAssign 9.3, ~~matrix multiplication~~ (cancelled)
- HW 2 Sec 9.3 #42, 44, A1, A2, A3 (submit for feedback in OneNote notebook, if desired)
- Proctorio check and all WebAssign to 9.3 due Sun
- Reading / Video assignment due Mon (cross product)
- (for yourself) Finish Worksheet 9.2, 9.3

9.3 Dot product

Motivation: What is the projection of $\vec{b}$ onto $\vec{a}$?

ex.) Draw the projection of $\vec{b}$ onto $\vec{a}$



Consider (1) What is the magnitude of each projection?
(2) When is $\text{proj}_{\vec{a}}\vec{b} = \vec{0}$?
(3) What are some real-life examples of projections?

The dot product of vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\cdot\vec{b}$ and can be computed as either

(i) $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ for $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ OR

(ii) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

Properties
- The dot product takes two vectors and gives a scalar.
- Vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b} = 0$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

Back to projections

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}}{|\vec{a}|^2}(\vec{a}\cdot\vec{b}) = \frac{\vec{a}}{|\vec{a}|}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)$$ vector! (the actual projection)

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = |\vec{b}|\cos\theta$$ scalar! (just the length and sign of the projection)

Monday August 31

Reminders

Quiz 1 tomorrow night (no check-in tomorrow)
Do WebAssign 9.4
Do HW 2, section 9.4 #26, 32, 35 for feedback
(Do homework on time, with academic integrity)

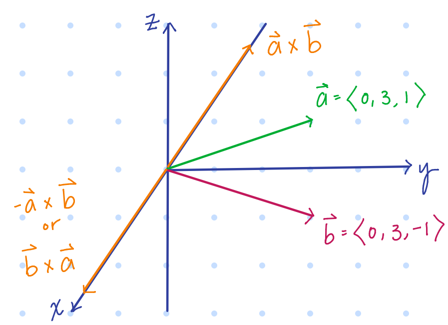
9.4 Cross product

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ be vectors in $\mathbb{R}^3$.
The cross product of $\vec{a}$ and $\vec{b}$, denoted $\vec{a} \times \vec{b}$, is given by

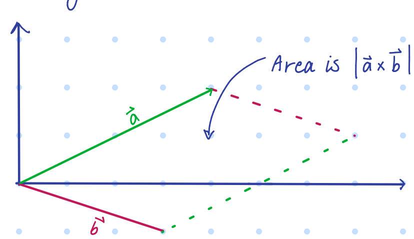
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Properties (aka the best tricks)

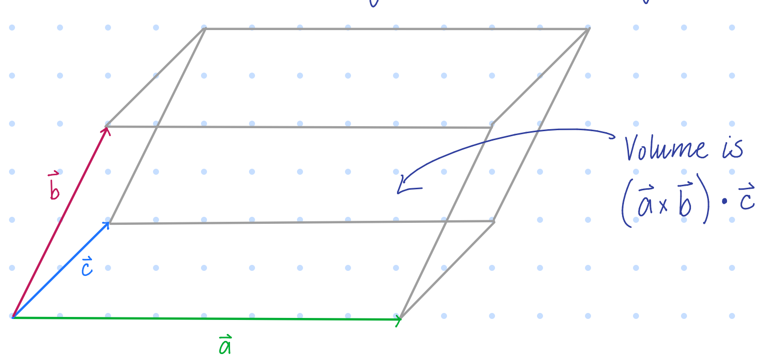
(i) Cross product gives a vector
(ii) $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$
(iii) The direction of $\vec{a} \times \vec{b}$ is determined by the right-hand rule
(iv) $|\vec{a} \times \vec{b}|$ is a scalar that represents the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$



(picture of (iii))



(v) $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is a scalar that gives the volume of the parallelepiped with edges $\vec{a}, \vec{b}, \vec{c}$

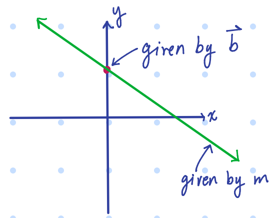


Ponder: If we "lower" $\vec{b}$ to the "ground", what happens to $(\vec{a} \times \vec{b}) \cdot \vec{c}$?
Answer: $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

9.5 Lines and Planes

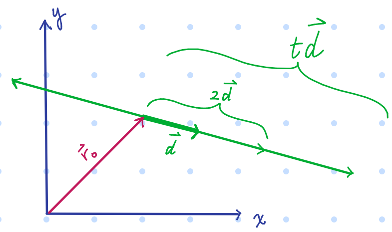
Lines

Old equation of a line : $y = \underset{\substack{\uparrow \\ \text{slope} \\ \text{"direction"}}}{m}x + \underset{\substack{\uparrow \\ \text{y-intercept} \\ \text{"one point"}}}{b}$



New (vector) equation of a line : $\vec{r} = \underset{\substack{\nearrow \\ \text{parameter}}}{t}\,\underset{\substack{\downarrow \\ \text{direction}}}{\vec{d}} + \underset{\substack{\downarrow \\ \text{point on line}}}{\vec{r}_0}$

(note: this equation is not unique)



ex) (a) Find an equation of a line through the point $(6,-5,2)$ and parallel to the vector $\langle 1, 3, -\frac{2}{3}\rangle$

$(6,-5,2)$ is a point on the line, so $\vec{r}_0 = \langle 6,-5,2\rangle$

The direction $\vec{v}$ is $\langle 1, 3, -\frac{2}{3}\rangle$.

The desired line is $\vec{r} = t\vec{v} + \vec{r}_0$

$$= t\langle 1, 3, -\tfrac{2}{3}\rangle + \langle 6, -5, 2\rangle$$

$$\boxed{= \langle 6 + t, \; -5 + 3t, \; 2 - \tfrac{2}{3}t \rangle}$$

Parametric form (awesome)

$$x = 6 + t$$
$$y = -5 + 3t$$
$$z = 2 - \tfrac{2}{3}t$$

$\uparrow \vec{r}_0$ $\uparrow \vec{d}$

Symmetric form (trash)

$$t = \frac{x-6}{1} = \frac{y--5}{3} = \frac{z-2}{-2/3}$$

(b) Find two other points on the line.

Pick any two values of t except zero. If $t=1$, we get $(7, -2, \frac{4}{3})$. If $t=3$, we get $(9, 4, 0)$

ex.) Find an equation of the line through point $(2,1,0)$ and perpendicular to both $\vec{v} = \langle 1, 2, -1\rangle$ and $\vec{w} = \langle 0, -4, -4\rangle$.

Compute $\vec{v}\times\vec{w}$ to find direction perpendicular to both of them

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 0 & -4 & -4 \end{vmatrix} = \langle 2(-4) - (-4)(-1), \; -(1(-4) - 0), \; 1(-4) - 0 \rangle = \langle -12, 4, -4 \rangle$$

Desired line: $\boxed{\vec{r} = \langle 2 - 12t, \; 1 + 4t, \; -4t \rangle}$

Tuesday, September 1

Reminders
- WebAssign 9.5
- Written HW, HW3 Sec 9.5 #16, 36, and A1
- Review conic sections
- ! Do Quiz 1 on Canvas

9.5 Lines and planes, cont.

Yesterday: equation of a line is $\vec{r} = t\vec{d} + \vec{r}_0$

ex3) Find an equation of the line segment through points $(6,1,3)$ and $(2,4,5)$

The direction vector $\vec{d}$ is formed by subtracting the two points. $\vec{d} = \langle 4,-3,-2 \rangle$
Pick one of the points to be $\vec{r}_0$. $\vec{r}_0 = \langle 6,1,3 \rangle$

Desired line segment: $\boxed{\vec{r} = \langle 6-4t, 1+3t, 3-2t \rangle \quad 0 \le t \le 1}$

ex4) Are lines L_1 and L_2 parallel, intersecting, or skew?
L_1: $x = t$, $y = 1+2t$, $z = 2+3t$
L_2: $x = 3-4s$, $y = 2-3s$, $z = 1+2s$

L_1 has direction vector $\langle 1,2,3 \rangle$
L_2 has direction vector $\langle -4,-3,2 \rangle$
The direction vectors are not parallel, so L_1 and L_2 are not parallel

To see if L_1 intersects L_2, try to find s and t so that $L_1(t) = L_2(s)$
If L_1 intersects L_2, then
$$\begin{cases} t = 3-4s \\ 1+2t = 2-3s \\ 2+3t = 1+2s \end{cases} \quad \text{for some } s,t$$

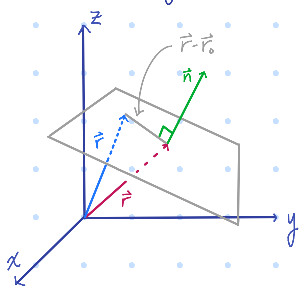
Solving, we get
$$\begin{aligned} 1+2(3-4s) &= 2-3s \\ 1+6-8s &= 2-3s \\ 5 &= 5s \\ s = 1&, \; t = -1 \end{aligned}$$

Plug t into L_1 and s into L_2 to see if $L_1(t) = L_2(s)$
$L_1(-1) = \langle -1,-1,-1 \rangle$, $L_2(1) = \langle -1,-1,3 \rangle$
L_1 and L_2 do not intersect, so they must be $\boxed{\text{skew}}$

Planes

We can't specify a plane exactly the same way that we specified a line

(Gesture wildly to explain why)



Vector equation of plane : $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$
where $\vec{n}$ is a normal vector, $\vec{r}_0$ is a point in the plane,
and $\vec{r}$ is the generic vector $\langle x,y,z\rangle$

Standard form of a plane: $ax+by+cz=d$
where $\langle a,b,c\rangle$ is a normal vector and d is a constant

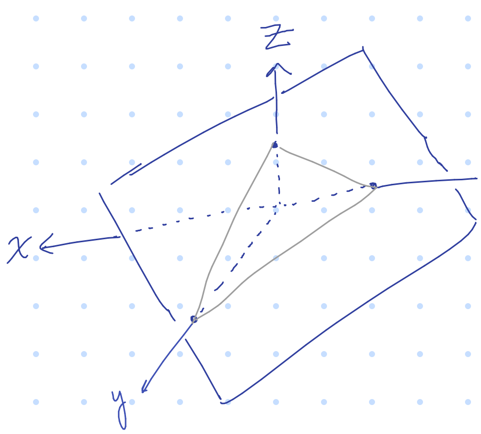
ex) Find an equation of the plane through the point $(6,3,2)$
and perpendicular to the vector $\langle -2,1,5\rangle$. Then find the intercepts and sketch.

$\vec{n}=\langle -2,1,5\rangle$
$\vec{r}_0=\langle 6,3,2\rangle$

Desired plane:
$$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$$
$$\langle -2,1,5\rangle\cdot(\langle x,y,z\rangle-\langle 6,3,2\rangle)=0$$
$$\langle -2,1,5\rangle\cdot\langle x-6,\ y-3,\ z-2\rangle=0$$
$$\boxed{-2(x-6)+(y-3)+5(z-2)=0}$$

To find the x-intercept, set y and z to zero. Similarly for y and z intercepts

x-int $(-\frac{1}{2},0,0)$	y-int $(0,1,0)$	z-int $(0,0,\frac{1}{5})$



ex1) Find an equation for the plane containing points $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$

Subtract points to find two vectors in the plane, and then take the cross product to get a normal vector. $\vec{v} = \langle -1, 1, 0 \rangle$, $\vec{u} = \langle 0, -1, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

Let $\vec{r}_0$ be $\langle 0,1,1 \rangle$. Then our desired plane is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$$\langle 1,1,1 \rangle \cdot (\langle x,y,z \rangle - \langle 0,1,1 \rangle) = 0$$
$$\langle 1,1,1 \rangle \cdot \langle x, y-1, z-1 \rangle = 0$$
$$x + y - 1 + z - 1 = 0$$
$$\boxed{x + y + z = 2}$$

ex2) Where does the line $x = 3-t$, $y = 2+t$, $z = 5t$ intersect the plane $x - y + 2z = 9$?

Find t so that $x(t), y(t), z(t)$ satisfy the plane equation.

$$(3-t) - (2+t) + 2(5t) = 9$$
$$8t = 8$$
$$t = 1$$

Find point given by $t=1$: $x(1) = 2$ $y(1) = 3$ $z(1) = 5$ $\boxed{\text{Point of intersection } (2,3,5)}$

ex3) How can you tell if 2 planes are parallel or intersecting?
Can they be skew? 🤔

ex4) Find distance between the point $(1,-2,4)$ and the plane $3x+2y+6z=5$

Plan:

Find vector between given point and a random point on the plane. Project it onto the normal vector of the plane.

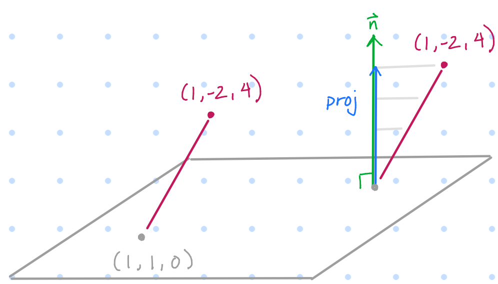
Computation:

$\vec{v} = \langle 1,-2,4\rangle - \langle 1,1,0\rangle = \langle 0,-3,4\rangle$

$\vec{n} = \langle 3,2,6\rangle$

$$\text{comp}_{\vec{n}}\vec{v} = \frac{\vec{n}\cdot\vec{v}}{|\vec{n}|} = \frac{(0\cdot 3)+(-3\cdot 2)+(4\cdot 6)}{\sqrt{3^2+2^2+6^2}} = \frac{18}{7}$$

Schematic:



ex5) Find distance between the point $(4,1,-2)$ and the line $x=1+t$, $y=3-2t$, $z=4-3t$

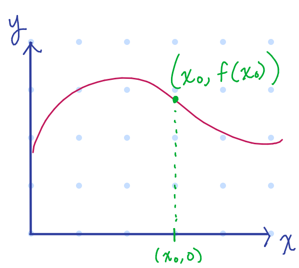
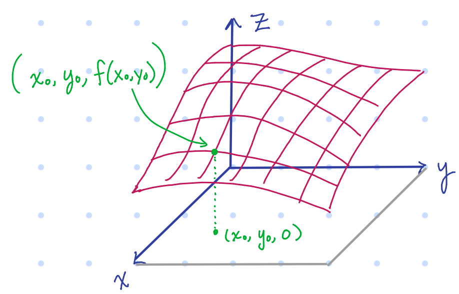
Hint: First draw a general picture and a plan like in example 4.

Wednesday, September 1

Reminders

- Quiz 1 corrections due tonight
- Compile HW 2 for André
- Finish "LaTex and 9.6 practice" (for Fri check-in)
- Review polar coordinates (for Fri)

9.6 Functions and surfaces

	Function of one variable	Function of two variables	Function of >2 vars
Examples	$f(x) = x\sin(x)$ $y = x^2$ (if interpreted as "y is a function of x")	$f(x,y) = xy\sin(x^2)$ $z = (x-1)^2 + (y-1)^2$ (if interpreted as "z is a function of x and y")	$f(x,y,z) = xyz + \ln(y)$ $S(r,\theta,h) = r\cos\theta + r\sin\theta$ $F(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$
Where to graph	$\mathbb{R}^2$ (xy plane)	$\mathbb{R}^3$ (3-space)	$\mathbb{R}^{n+1}$
Example graph			

A **z-trace** or cross-section of a surface $z=f(x,y)$ is the intersection of the surface with the plane $z=k$ for some constant k

y-traces and x-traces also exist.

ex 1) Draw the traces of $f(x,y)=6-3x-2y$ for $z=0,3,6,9$. Sketch $f(x,y)$.

$\underline{z=0}$

$0=6-3x-2y$

$y=-\frac{3}{2}x+3$

$\underline{z=3}$

$3=6-3x-2y$

$y=-\frac{3}{2}x+\frac{3}{2}$

$\underline{z=6}$

$6=6-3x-2y$

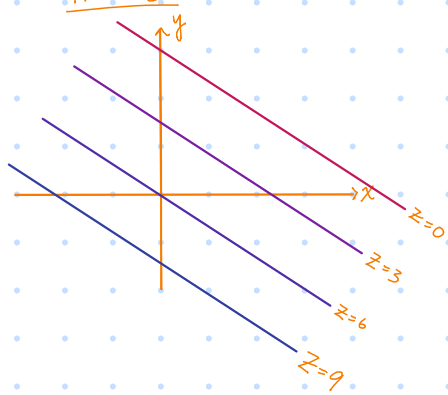
$y=-\frac{3}{2}x$

$\underline{z=9}$

$9=6-3x-2y$

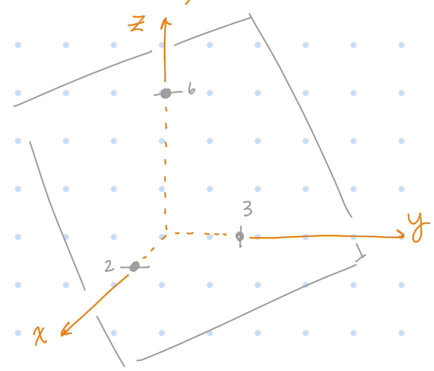
$y=-\frac{3}{2}x-\frac{3}{2}$

Traces



Sketch

intercepts (0,0,6), (0,3,0), (2,0,0)



ex 2) Draw the traces of $f(x,y)=x^2$ for $z=-1,0,1,2$. Sketch $f(x,y)$.

ex3) Draw the traces of $f(x,y)=4x^2+y^2$ for $z=0,2,4,6$. Sketch $f(x,y)$.

Friday, September 3

Reminders

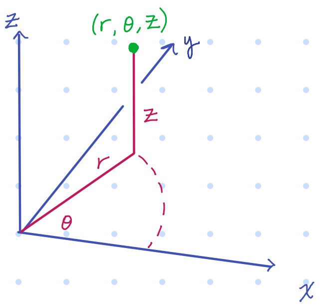
- WebAssign 9.6, 9.7 (everything up to 9.7 due Sunday)
- Written HW 3: Sec 9.6 #15, 27
  Sec 9.7 #26 (assume $r \geq 0$), 28, 32
  A2 (one surprising one!), A3
- Memorize 6 quadric surfaces (9.6 Table 2)

9.7 Cylindrical and spherical coordinates

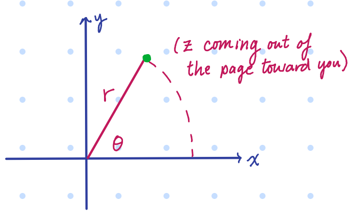
Cylindrical coordinates

Drawing of cylindrical coordinates

(3D view)



(Top view)



Relationship with Cartesian

Cartesian (x, y, z) → Cylindrical (r, θ, z):

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$
$$z = z$$

Cylindrical (r, θ, z) → Cartesian (x, y, z):

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

ex 1) What is the shape described in cylindrical coordinates? Sketch it

(a) $r = 3$

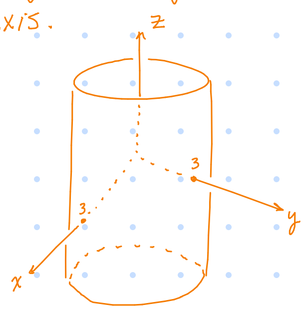
(b) $\theta = \pi/4$

(c) $z = r^2$

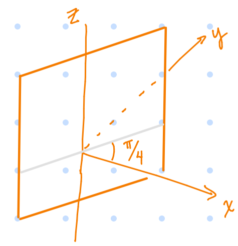
(d) $4 \le r \le 5$, $0 \le \theta \le \pi$, $0 \le z \le 1$

(e) $r < 1$, $0 \le \theta \le \pi/4$, $0 \le z \le 1$

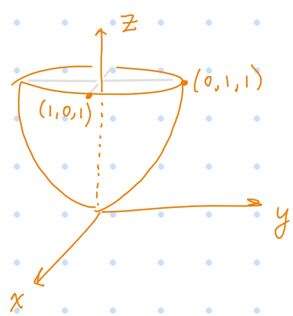
(a) infinite cylinder of radius 3 along z-axis.



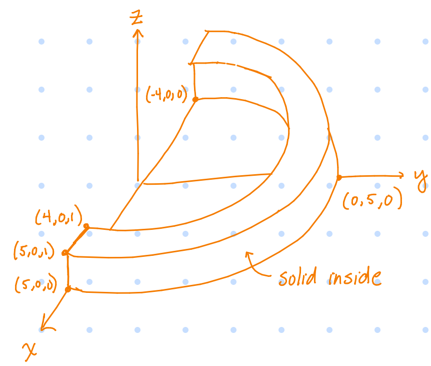
(b) plane at an angle



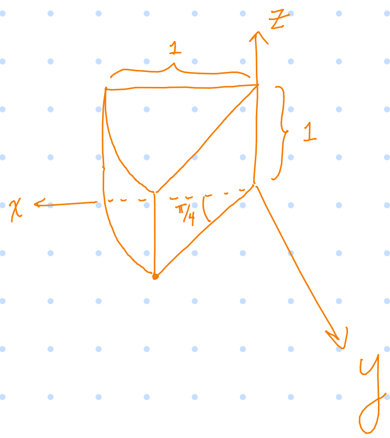
(c) paraboloid along z-axis



(d) a "C"-shaped solid

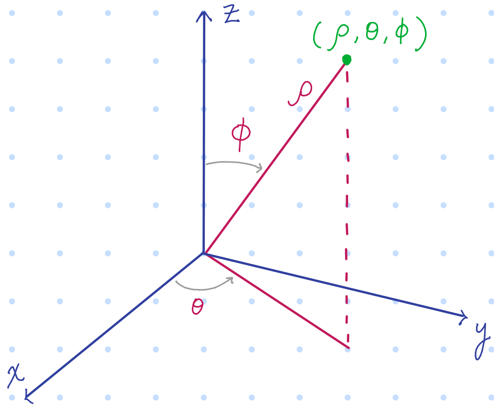


(e) solid wedge of cheese



Spherical coordinates

Drawing of spherical coordinates



Relationship with Cartesian

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan\theta = \frac{y}{x}$$

$$\begin{matrix}\text{Cartesian} \\ (x, y, z)\end{matrix} \;\rightleftarrows\; \begin{matrix}\text{Spherical} \\ (\rho, \theta, \phi)\end{matrix}$$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

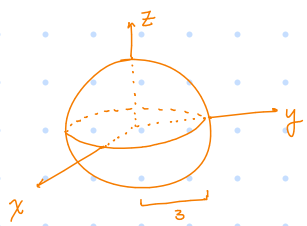
Greek letter phi ("fee")

Greek letter rho ("roe")

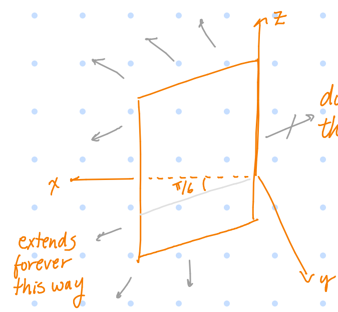
ex2) What is this shape described in spherical coordinates? Sketch it.

(a) $\rho = 3$
(b) $\theta = \pi/6$
(c) $\phi = \pi/3$
(d) $4 \le \rho \le 5$, $0 \le \phi \le \pi/2$
(e) $\rho \le 3$, $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$
(f) $\rho \le 3$, $\frac{3\pi}{4} \le \phi \le \pi$

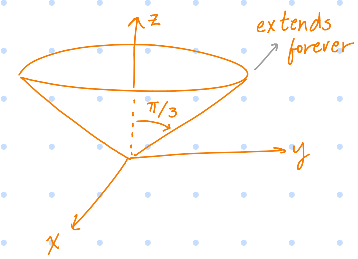
(a) sphere of radius 3



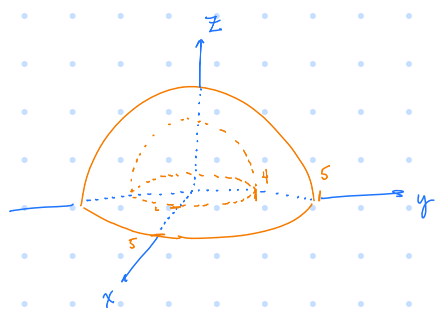
(b) half plane



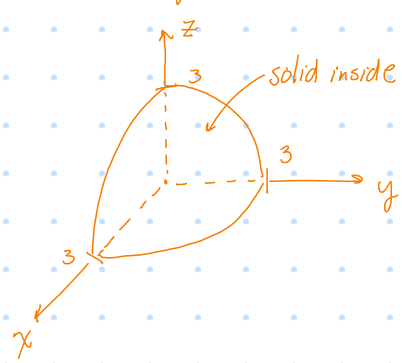
(c) infinite cone



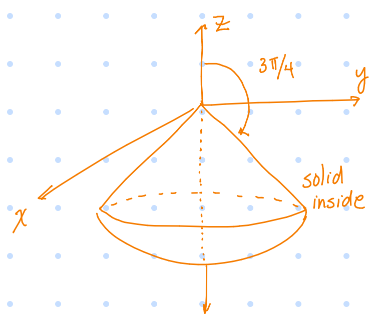
(d) solid half dome with thickness 1



(e) the part of the solid sphere of radius 3 in the first octant



(f) solid cone with round cap.



Tuesday, September 8

Reminders

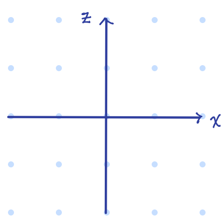
- Think about groupmates for group project
- Review parametric equations

Today's activity: Go to student.desmos.com and use class code QDCAC8
Make sets of 3 cards by matching one 3D graph, one contour plot, and one equation.

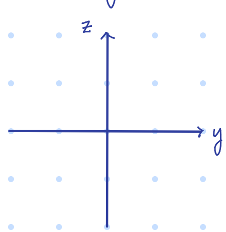
Check-in 5 (due in class on Sept 8)

Consider the quadric surface given by $x^2 + z^2 = y^2 + 4$

a.) Draw the trace of this surface at $y=0$



b.) Draw the trace of this surface at $x=0$



c.) What is the shape of this quadric surface? Sketch it.

<u>Wednesday, September 9</u>

Reminders

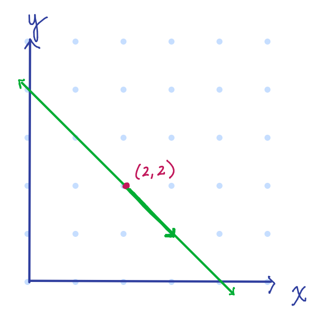
- WebAssign 10.1
- HW 4 ...?
- Meet group + pick topic for graphing project (due Thur)
- Prep for check-in 6

10.1 Vector functions + space curves

A vector-valued function is a function with codomain $\mathbb{R}^n$. In other other words, the output is a vector
The graph of a vector-valued function is the set of all points at the tip of the vector output

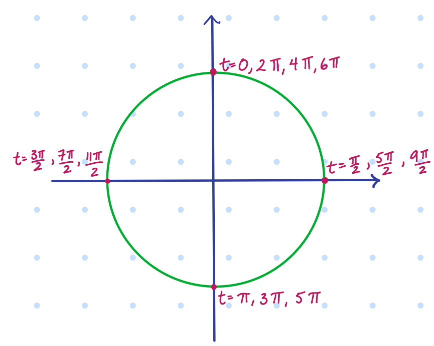
ex 1) $\vec{r}(t) = \langle \underbrace{2+t}_{x(t)}, \underbrace{3-t}_{y(t)} \rangle$

This is a vector function from $\mathbb{R}$ to $\mathbb{R}^2$.
Its graph is a line containing the point $(2,3)$ in the direction $\langle 1,-1 \rangle$



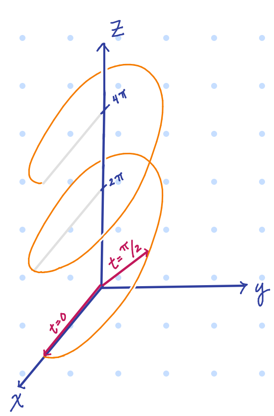
ex 2) $\vec{r}(t) = \langle \sin(t), \cos(t) \rangle \quad 0 \le t \le 6\pi$

This is a vector function from $\mathbb{R}$ to $\mathbb{R}^2$.
Its graph is a circle of radius 1, starting at $(0,1)$ and covering the circle three times clockwise.



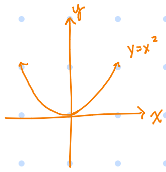
ex 3) $\vec{r}(t) = \langle \underbrace{\cos(t)}_{x(t)}, \underbrace{\sin(t)}_{y(t)}, \underbrace{t}_{z(t)} \rangle \quad 0 \le t \le 4\pi$

This is a vector function from $\mathbb{R}$ to $\mathbb{R}^3$.
Its graph is a helix starting at $(1,0,0)$ and ending at $(1, 0, 4\pi)$

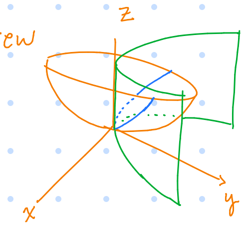


ex 4) Find a vector function that represents the curve of intersection between $z = 4x^2 + y^2$ and $y = x^2$

Top view

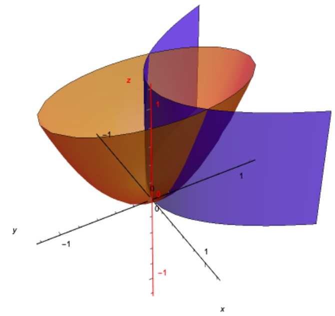


3D view



I tried

Mathematica's plot



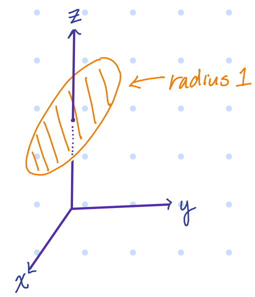
$$\boxed{\vec{r}(t) = \langle\, t \,,\; t^2 \,,\; 4t^2 + t^4 \,\rangle}$$

we know the curve lies on $z = 4x^2 + y^2$, so z must be

$$4\underset{x}{(t)^2} + \underset{y}{(t^2)^2}$$

ex 5) $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2\rangle$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

This is a vector function from $\mathbb{R}^2$ to $\mathbb{R}^3$. Its graph is a disc of radius 1 in the plane $z=2$ centered at $(0,0,2)$.



ex 6) What is the dimension of each graph in examples 1-5?

examples 1-4 : one dimension

example 5 : two dimensions

Computations with vector-valued functions

Given a vector-valued function, we can compute the following things component-wise:

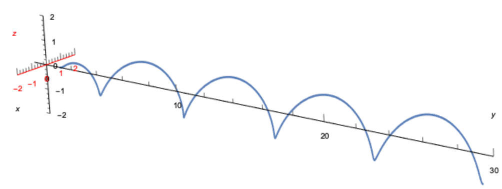
(a) Domain

ex) The domain of $\vec{r}(t) = \langle \underbrace{t^2}_{\text{no restriction}}, \underbrace{\ln(t)}_{\text{t must be greater than 0}}, \underbrace{\sqrt{3-t}}_{t \text{ must be } \leq 3} \rangle$ is $0 < t \leq 3$

(b) Limit

ex) $\vec{r}(t) = \langle e^{-1/t}\cos(t),\ t+1,\ e^{-1/t}\sin(t) \rangle \qquad 0 < t \leq 30$

$\lim_{t \to 0^+} \vec{r}(t) = \langle 0, 1, 0 \rangle$



(c) Derivative

ex) $\vec{r}(t) = \langle \ln(t),\ t^3+4,\ \arctan(t) \rangle$

$\vec{r}\,'(t) = \langle \frac{1}{t},\ 3t^2,\ \frac{1}{1+t^2} \rangle$

(d) Intersection of curves

ex) If particle A travels along path $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and particle B travels along path $\vec{p}(s) = \langle x(s), y(s), z(s) \rangle$,

(a) do their paths cross? If we can find some t_0 and s_0 so that $\vec{r}(t_0) = \vec{p}(s_0)$, then yes.

(b) do the particles collide? If we can find some t_0 and s_0 so that $\vec{r}(t_0) = \vec{p}(s_0)$ and $t_0 = s_0$, then yes.

Friday, September 11

Reminders

- WebAssign 10.2 . Week 3 WebAssign due Sun
- Written HW 4: 10.1 #6, 25 ; 10.2 #2, 26, 32, 38 ; A1, A2
- Review for Quiz 2
- Think about group graphing project

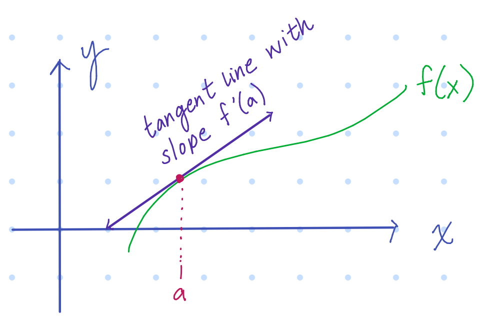
10.2 Derivatives and integrals of vector functions

The derivative of a vector function $\vec{r}(t)$ is denoted $\vec{r}\,'(t)$ or $\frac{d\vec{r}}{dt}$. It is computed by applying $\frac{d}{dt}$ to each component. That is, if $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

Derivative $f'(x)$ in Calc 1

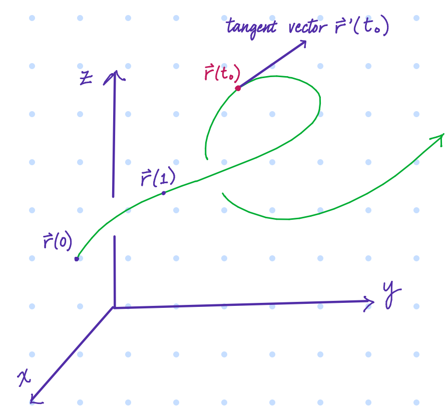
- f' is a scalar
- $f'(a)$ is the slope of the tangent line at $x=a$



- If $f(x)$ is the position of a particle at time x, then $f'(x)$ is its velocity
- If $f'(a)=0$, then $f(x)$ has a horizontal tangent line at $x=a$.
- For a given "picture" of a curve, the derivative at a specific point is unique

Derivative $\vec{r}\,'(t)$ in Calc 3

- $\vec{r}\,'(t)$ is a vector
- $\vec{r}\,'(t_0)$ is a tangent vector of the graph of $\vec{r}(t)$ at the point $\vec{r}(t_0)$



- If $\vec{r}(t)$ is the position of a particle at time t, then $\vec{r}\,'(t)$ is the velocity vector
- If $\vec{r}\,'(t_0)=0$, then the path "pauses" at $t=t_0$
- For a given "picture" of a curve, the derivative at a specific point depends on the specific choice of parametrization

The *unit tangent vector* of $\vec{r}(t)$ is denoted $\vec{T}(t)$. We compute $\vec{T}(t)$ by dividing the derivative by its magnitude.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|r'(t)|}$$

ex 1) Give an equation of the tangent line of $\vec{r}(t) = \langle e^t,\ te^t,\ te^{t^2} \rangle$ at $(1,0,0)$

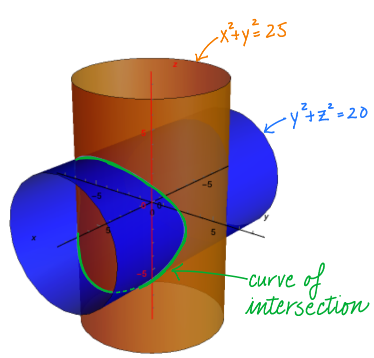
The point $(1,0,0)$ occurs on $\vec{r}(t)$ when $t=0$.

The direction vector of the tangent line at $(1,0,0)$ is $\vec{r}\,'(0)$

$$\vec{r}\,'(t) = \langle e^t,\ te^t + e^t,\ t\cdot 2te^{t^2} + e^{t^2} \rangle$$
$$= \langle e^t,\ e^t(t+1),\ e^{t^2}(2t^2+1) \rangle$$
$$\vec{r}\,'(0) = \langle 1,\ 1,\ 1 \rangle$$

The tangent line is $\vec{\ell}(t) = \langle 1,0,0 \rangle + t\langle 1,1,1 \rangle$
$= \langle 1+t,\ t,\ t \rangle$

ex 2) Give an equation of the tangent line of the curve of intersection of the cylinders $x^2+y^2=25$ and $y^2+z^2=20$ at the point $(3,4,2)$



1) parametrize curve of intersection

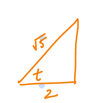
projection on yz-plane is a circle of radius $\sqrt{20}$,

so let $y=\sqrt{20}\cos t$, $z=\sqrt{20}\sin t$ for $0 \le t \le 2\pi$.

Then the curve lies on $x^2+y^2=25$, so $x = \sqrt{y^2-25} = \sqrt{25-20\cos^2 t}$.

Curve: $\vec{r}(t) = \langle \sqrt{25-20\cos^2 t},\ \sqrt{20}\cos t,\ \sqrt{20}\sin t \rangle$

Point $(3,4,2)$ occurs at $t = \arcsin\left(\frac{1}{\sqrt{5}}\right)$



2) Find tangent line

$$\vec{r}\,'(t) = \langle \tfrac{1}{2}(25-20\cos^2 t)^{-\frac{1}{2}}(40\cos t\sin t),\ -\sqrt{20}\sin t,\ \sqrt{20}\cos t \rangle$$
$$\vec{r}\,'\left(\arcsin\left(\tfrac{1}{\sqrt{5}}\right)\right) = \langle 8/3,\ -2,\ 4 \rangle$$

Note $\langle 4,-3,6 \rangle$ is also a good direction vector.

$$\vec{\ell}(t) = \langle 3+4t,\ 4-3t,\ 2+6t \rangle$$

3 more facts

- $\int \vec{r}(t)\,dt = \langle \int x(t)\,dt,\ \int y(t)\,dt,\ \int z(t)\,dt \rangle$ (remember 3 separate "+C")
- Product rules
  (a) $\frac{d}{dt}[f(t)\vec{v}(t)] = f'(t)\vec{v}(t) + f(t)\vec{v}\,'(t)$
  (b) $\frac{d}{dt}[\vec{v}(t)\cdot\vec{u}(t)] = \vec{v}\,'(t)\cdot\vec{u}(t) + \vec{v}(t)\cdot\vec{u}\,'(t)$
  (c) $\frac{d}{dt}[\vec{v}(t)\times\vec{u}(t)] = \vec{v}\,'(t)\times\vec{u}(t) + \vec{v}(t)\times\vec{u}\,'(t)$
- Chain rule $\frac{d}{dt}[\vec{u}(f(t))] = \vec{u}\,'(f(t))\,f'(t)$

Check-in 6 (due in class Sept 11)

Find a vector equation of the curve of intersection of the cylinder $x^2+y^2=25$ and the plane $x+y+z=7$

Monday September 13

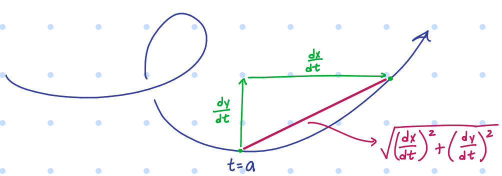
Reminders

- WebAssign 10.3
- HW 4 section 10.3 #14, 15
- Study for Quiz 2

10.3 Arc length

If you have a parametrized curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, how do you compute the length of the curve from $t=a$ to $t=b$?

Notice $x'(t) = \frac{dx}{dt}$ can be interpreted as "change in x per unit of t (maybe time)"



We can use the Pythagorean Theorem / distance formula to compute the length of the segment that approximates the curve.

"Add" all the (infinitesimal) segment lengths to get the whole length.

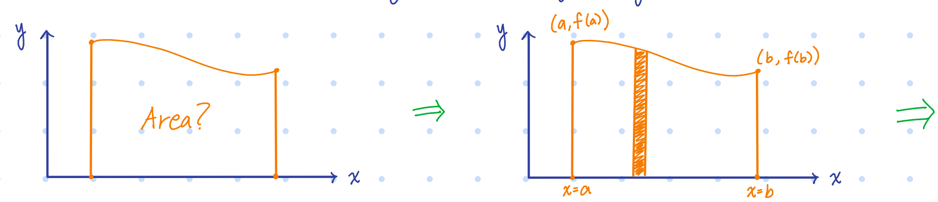
$$\boxed{\begin{array}{l}\text{Arc length formula of curve } \vec{r}(t) \text{ from } t=a \text{ to } t=b \\ L = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt\end{array}}$$

... or $\quad L = \int_a^b |\vec{r}\,'(t)|\, dt$

Q: What is a parametrization, really?

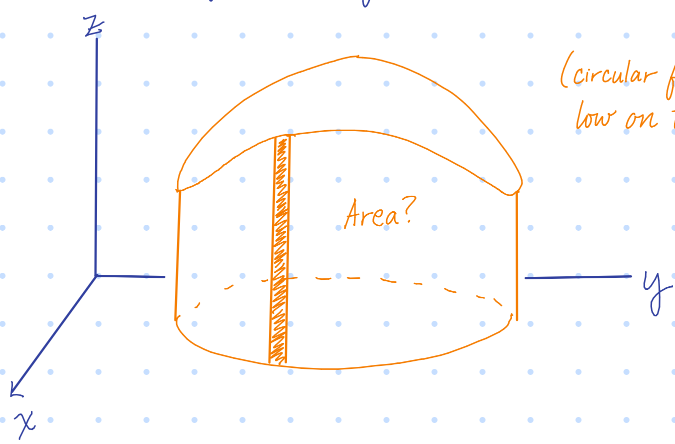
A: It is a choice of how we want to place a grid on a shape.

In Calc 1 and 2, a function always used the grid system on the x-axis.



$\Rightarrow$ Area $= \int_a^b f(x)\,dx$

In Calc 3, a function may not have an obvious grid



(circular fence that's low on the left + right)

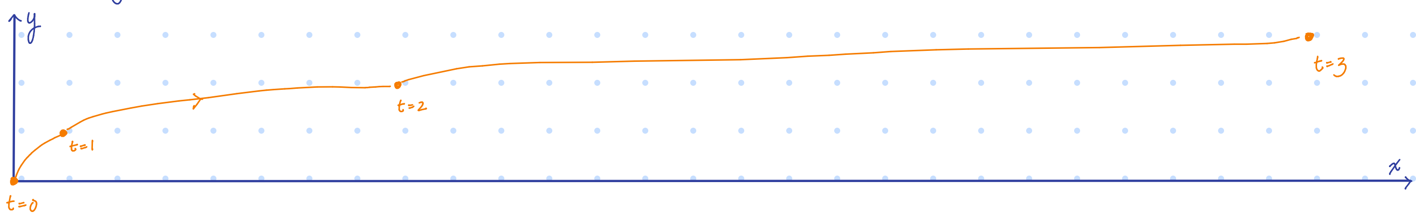
In order to find the area of the fence, I need to pick where to start, which direction to go, and when to end.

That's what a parametrization is!

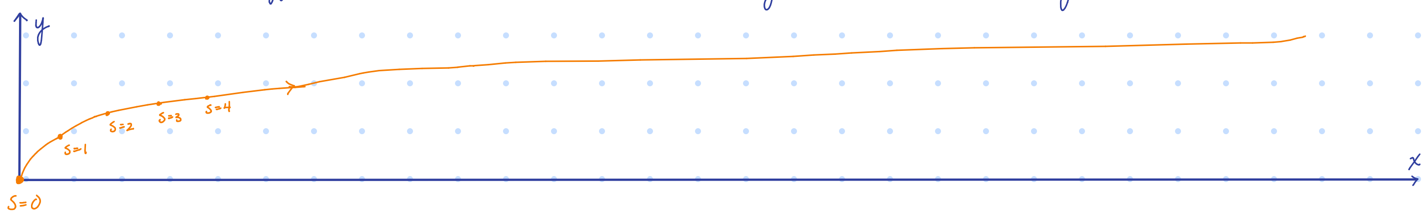
Reparametrizing by arc length

Consider $\vec{r}(t) = \langle t^3, t \rangle$

Here is a graph that shows its current parametrization.



The points for consecutive values of t vary a lot in distance!
We can choose a different parametrization that's evenly spaced wrt arc length, like this.



Here's how:

1) Compute the arc length from a fixed point $\vec{r}(a)$ to an arbitrary point $\vec{r}(t)$.
This is called the arc length function $s(t)$.

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

2.) Take $s(t)$ and solve for t so you have $t(s)$.

3.) Plug $t(s)$ into $\vec{r}(t)$ to get $\vec{r}(t(s))$. Now $\vec{r}$ is given in terms of arc length s!

ex 1) Reparametrize $\vec{r}(t) = \langle 3\sin t, 3\cos t, t \rangle$, $t \geq 0$, by its arc length measured from $(0,3,0)$.

step 1) $\vec{r}\,'(t) = \langle 3\cos t, -3\sin t, 1 \rangle$

The point $(0,3,0)$ occurs at $t=0$.

Find arc length function $s(t) = \int_0^t \sqrt{(3\cos u)^2 + (-3\sin u)^2 + 1^2}\, du$

$$= \int_0^t \sqrt{10}\, du$$

$$= u\sqrt{10}\,\Big]_0^t$$

$$s(t) = t\sqrt{10}$$

step 2) Since $s(t) = t\sqrt{10}$, we solve for t to get $t(s) = \frac{s}{\sqrt{10}}$

step 3) Plug $t(s)$ into $\vec{r}(t)$. $\boxed{\vec{r}(s) = \left\langle 3\sin\left(\frac{s}{\sqrt{10}}\right), 3\cos\left(\frac{s}{\sqrt{10}}\right), \frac{s}{\sqrt{10}} \right\rangle, \quad s \geq 0}$

ex 2) What is the arc length of the curve in example 1 from $(0,3,0)$ to $\left(3\sin\left(\frac{7}{\sqrt{10}}\right), 3\cos\left(\frac{7}{\sqrt{10}}\right), \frac{7}{\sqrt{10}}\right)$? $\boxed{7}$

Tuesday, September 15

Reminders

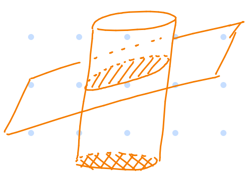
- ! Quiz 2 tonight !

Go to student.desmos.com, use code QDC AC8 and do the Parametric Matching activity.

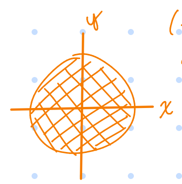
10.5 Parametric surfaces

ex 1) Give a vector function for the portion of $x+y+3z=3$ contained inside $x^2+y^2=49$.

1) Sketch



2) Projection



(filled-in disc of radius 7)

3) Parametrize projection to get x and y coordinates

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ \_\_\_\_\_\_ \rangle$$
$$0 \le v \le 2\pi$$
$$0 \le u \le 7$$

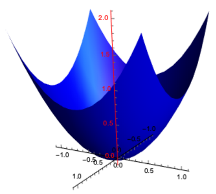
4) Use the fact that our shape lies on $x+y+3z=3$ to find z-coordinate.

$$3z = 3 - x - y$$
$$z = \tfrac{1}{3}(3 - x - y)$$
$$z = \tfrac{1}{3}(3 - u\cos v - u\sin v)$$

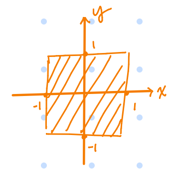
$$\boxed{\begin{gathered}\vec{r}(u,v) = \left\langle u\cos v,\ u\sin v,\ \tfrac{1}{3}(3 - u\cos v - u\sin v) \right\rangle \\ 0 \le v \le 2\pi \\ 0 \le u \le 7\end{gathered}}$$

ex 2) Give a parametrization for the portion of $z = x^2 + y^2$ satisfying $-1 \le x \le 1$, $-1 \le y \le 1$.

1) Sketch



2.) Find convenient projection



3.) Use projection to get x and y coordinates

$\vec{r}(u,v) = \langle u, v, \_\_\_\_\_ \rangle$

$-1 \le u \le 1$

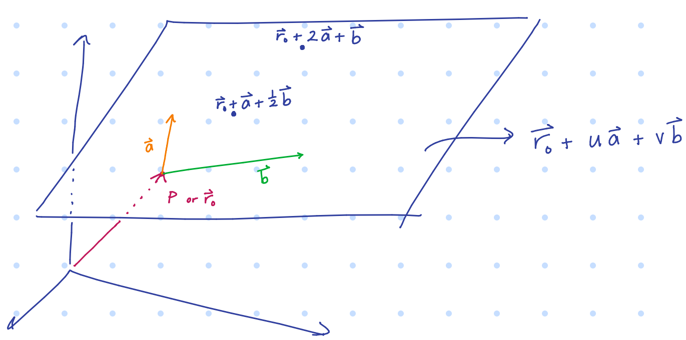
$-1 \le v \le 1$

4) Get z-coordinate by lifting $(x,y,0)$ onto surface.

$$z = x^2 + y^2$$
$$= u^2 + v^2$$

$$\boxed{\begin{aligned} &\vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle \\ &-1 \le u \le 1 \\ &-1 \le v \le 1 \end{aligned}}$$

How to parametrize plane with point $P(p_1, p_2, p_3)$ and vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$.



$$\boxed{\begin{aligned} &\text{vector equation of a plane} \\ &\vec{r}(u,v) = \langle p_1, p_2, p_3 \rangle + u\langle a_1, a_2, a_3 \rangle + v\langle b_1, b_2, b_3 \rangle \\ &-\infty < u < \infty \\ &-\infty < v < \infty \end{aligned}}$$

ex 3) Give a vector equation of the plane through point $(1, 2, -3)$ and containing $\langle 1, 1, -1 \rangle$ and $\langle 1, -1, 1 \rangle$.

$$\vec{r}(u,v) = \langle 1, 2, 3 \rangle + u\langle 1, 1, -1 \rangle + v\langle 1, -1, 1 \rangle$$

$$\vec{r}(u,v) = \langle 1+u+v, \ 2+u-v, \ 3-u+v \rangle$$

$-\infty \le u \le \infty$

$-\infty \le v \le \infty$

Wednesday, September 16

Reminders

- Compile HW 4 for André
- WebAssign 10.5
- Quiz corrections due 10 pm tonight

10.5 Surfaces (cont)

ex4) Give a vector equation of the sphere of radius 2 centered at the origin.

Spheres are nice in spherical coordinates! In spherical coordinates, this shape would be

$$\rho = 2 \quad , \quad 0 \le \theta \le 2\pi \quad , \quad 0 \le \phi \le \pi$$

Now use the spherical-to-rectangular conversion formulas to get our answer.

Formulas

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$$\Longrightarrow \boxed{\vec{r}(u,v): \quad \begin{aligned} x &= 2\sin u \cos v \\ y &= 2 \sin u \sin v \\ z &= 2\cos u \end{aligned} \qquad \begin{aligned} 0 \le u \le \pi \\ 0 \le v \le 2\pi \end{aligned}}$$

ex5) Find 2 different parametrizations for the upper half of the sphere of radius 2

parametrization 1 : $\vec{r}(u,v) = \langle 2\sin u \cos v \, , \, 2 \sin u \sin v \, , \, 2\cos u \rangle$

$0 \le u \le \pi/2$

$0 \le v \le 2\pi$

Same as ex4 but domain of ϕ is reduced

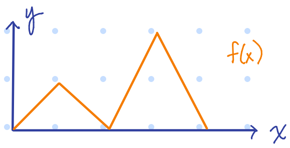
parametrization 2 : $\vec{r}(u,v) = \langle u\cos v, \, u \sin v, \, \sqrt{25 - u^2} \rangle$

$0 \le u \le 2$

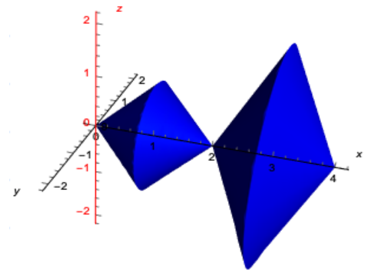
$0 \le v \le 2\pi$

using projection and $x^2 + y^2 + z^2 = 4$

Shortcut for parametrizing surfaces with cross-sections that are circles (a.k.a. surfaces of revolution)



spin!

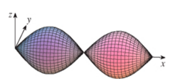


If a surface is generated by "spinning" the 2D graph $y = f(x)$ around a third axis, then one possible parametrization is

$$\boxed{\vec{r}(x,\theta) = \langle\, x \;,\; f(x)\cos\theta \;,\; f(x)\sin\theta \,\rangle \qquad 0 \le \theta \le 2\pi \;,\; x \text{ whatever it is in the graph of } f(x)}$$

ex 6) Find a vector equation that generates a graph like this :

Figure 11



$\vec{r}(u,v) = \langle\, u \,,\, \sin(u)\cos(v) \,,\, \sin(u)\sin(v) \,\rangle \;,\; 0 \le u \le 2\pi \;,\; 0 \le v \le 2\pi$

in Mathematica:

```
ParametricPlot3D [ { u, Sin[u] Cos[v], Sin[u] Sin[v] }, {u, 0, 2Pi}, {v, 0, 2Pi} ]
```

Group Members: ____

Timeline for Project

- Sept 16: Project assigned
- Sept 17: Project proposals due via email by 11:59 PM
- Sept 18: Work on project in class and present progress as Check-in 7
- Sept 25: Work on project in class and present progress as Check-in 9
- Oct 6: Project and reflections due via upload to Canvas as Check-in 10 (Upload a .NB file of your Mathematica notebook and a .PDF of your reflection questions. The grade will count toward Check-ins 10, 11, and 12)

Grading Rubric (24 points total)

- (+6 pts) There are at least 12 parametric plots
- (+5 pts) At least five of the equations are different ~~quadric~~ plane surfaces (ellipse, paraboloid, hyperboloid, etc)
- (+5 pts) At least five of the equations are different curves
- (+4 pts) Answer the reflection questions
- (+4 pts) Pledge below is signed by all group members

Bonus points

- (+3 pts) Math department favorite
- (+3 pts) Most diverse set of equations
- (+3 pts) Most difficult equation
- (+3 pts) (creative?)

Pledge: I certify that every group member contributed meaningfully to this project

______________________ Signature ______________________ Signature

______________________ Signature ______________________ Signature

Please complete these reflection questions together after completing your project.

1. Which part of your graph was the most difficult to make? What made it so difficult? How did your team eventually figure it out?

2. What was your most meaningful contribution to the project?

3. What is one thing you learned about parametrizing shapes from working on this?

Friday September 18

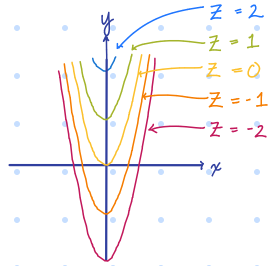
Reminders

- WebAssign 11.1, and week 4 WebAssign due Sun
- Written HW 5: everything except 11.2. [Kinda long, start early]
- Work on group project a bit ☺
- Review limit defn of continuity (links in Piazza)

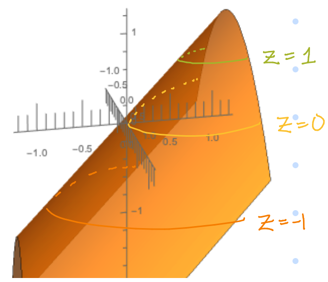
Quick facts for 11.1 WebAssign and HW.

1.) A contour map is a picture of several z-traces drawn on the same axes.

ex. Contour map of function $f(x,y) = y - x^2$



Actual graph



2) Graph transformations of single variable functions works the same for multivariable functions.

Given $y = f(x)$...		Given $z = f(x,y)$...	
• $f(x) + 5$	shifts the graph 5 units in the positive y-direction	• $f(x,y) + 5$	shifts the graph 5 units in the positive z-direction
• $f(x+5)$	shifts the graph 5 units in the negative x-direction	• $f(x+5, y)$	shifts the graph 5 units in the negative x-direction
		$f(x, y+5)$	shifts the graph 5 units in the negative y-direction
• $-f(x)$	reflects the graph over the x-axis (turn y into -y)	• $-f(x,y)$	reflects the graph over the xy-plane (turn z into -z)
• $f(-x)$	reflects the graph over the y-axis (turn x into -x)	• $f(-x,y)$	reflects the graph over the yz-plane (turn x into -x)
		$f(x,-y)$	reflects the graph over the xz-plane (turn y into -y)
• $3f(x)$	stretches the graph by a factor of 3 in y-direction	• $3f(x,y)$	stretches the graph by a factor of 3 in z-direction
• $f(3x)$	compresses the graph by a factor of 3 in x-direction	• $f(3x,y)$	compresses the graph by a factor of 3 in x-direction
		$f(x,3y)$	compresses the graph by a factor of 3 in y-direction

Monday September 21

Reminder

- Work on graphing project

11.1 Functions of several variables (cont)

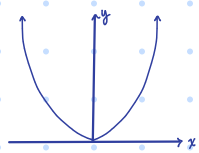
A scalar-valued function can have however many variables we want! But the graphs will become hard to draw because the dimension will be very high. We will now discuss how to think about these functions of many variables.

Notation

function of 1 variable: $y = f(x)$
function of 2 vars: $z = f(x,y)$
function of 3 vars: $w = f(x,y,z)$ (since w comes after x, y, z, of course ...)
function of n vars: $w = f(x_1, \dots, x_n)$

Graphs

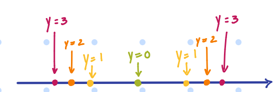
function of 1 var has a graph of dimension 1 that lives in $\mathbb{R}^2$



Graph of $f(x) = x^2$

Level sets (a.k.a. traces of the output var)

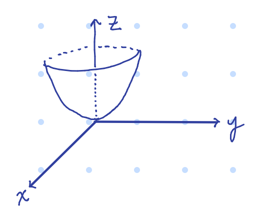
... and has level sets of dimension 0 drawn in $\mathbb{R}^1$. (We never actually draw these, but I'll draw one here for the sake of analogy)



Level sets of $f(x) = x^2$

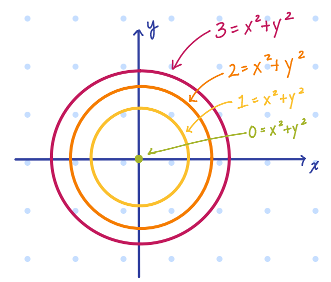
---

function of 2 vars have a graph of dimension 2 that lives in $\mathbb{R}^3$



Graph of $f(x,y) = x^2 + y^2$

... and have level sets of dimension 1 drawn in $\mathbb{R}^2$. (Also called level curves because they're curves)



Level sets of $f(x,y) = x^2 + y^2$
(concentric circles)

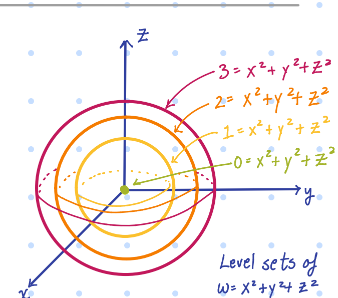
---

function of 3 vars have a graph of dimension 3 that lives in $\mathbb{R}^4$

[??]

Graph of $w = x^2 + y^2 + z^2$
(Don't know how to draw $\mathbb{R}^4$!)

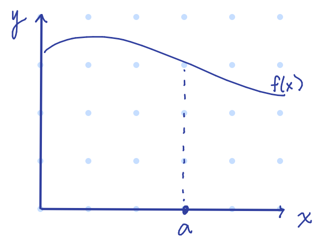
... and have level sets of dimension 2 drawn in $\mathbb{R}^3$. (Also called level surfaces because they're surfaces)



Level sets of $w = x^2+y^2+z^2$
(concentric spheres)

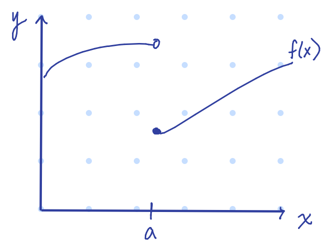
11.2 Limits and continuity

When we worked with functions of 1 variable in Calc 1, we knew that a limit existed if the limits from the left and right both existed and were equal to each other.



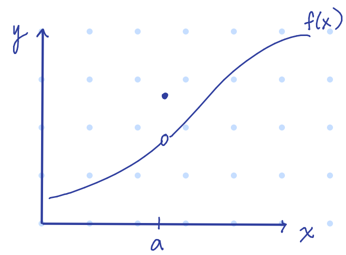
Does $\lim_{x\to a} f(x)$ exist?

yes / no



Does $\lim_{x\to a} f(x)$ exist?

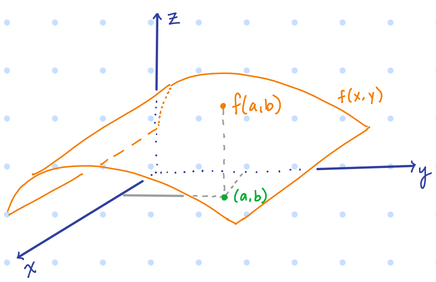
yes / no



Does $\lim_{x\to a} f(x)$ exist?

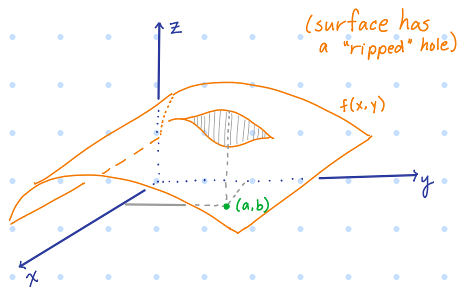
yes / no

The same idea works for functions of several variables, but now instead of approaching a point from the left and right, there are infinitely many ways to approach the point!



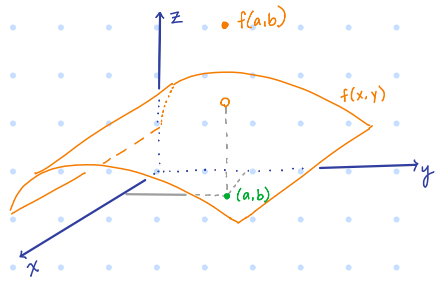
Does $\lim_{(x,y)\to(a,b)} f(x,y)$ exist?

yes / no



Does $\lim_{(x,y)\to(a,b)} f(x,y)$ exist?

yes / no



Does $\lim_{(x,y)\to(a,b)} f(x,y)$ exist?

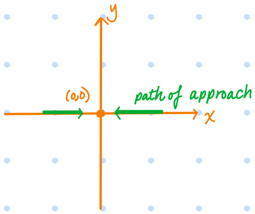
yes / no

Here's how we actually compute limits algebraically

ex 1) Compute $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Try to approach (0,0) along x-axis.

(top view)



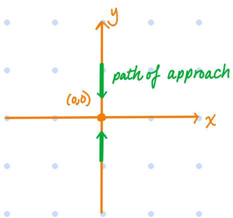
The path of approach contains points whose y-coordinates are all zero, so set $y=0$, and this will reduce the problem to the limit of a 1-variable function

$$\lim_{x\to 0} \frac{x\cdot 0}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

[Note: This 1-var limit does not require l'Hôpital's rule because the numerator is truly, exactly zero and not just going to zero as a limit.]

Try to approach (0,0) along y-axis.

(top view)



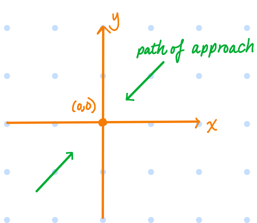
[similar to approaching along x-axis]

$$\lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Same as before!

Try to approach (0,0) along the line $y=x$

(top view)



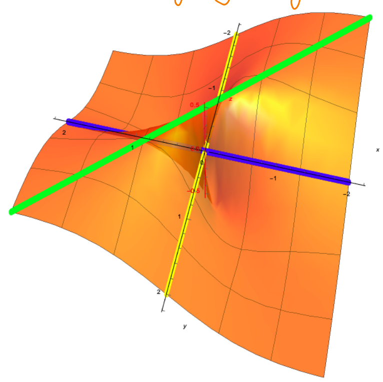
Every point along the line $y=x$ has identical x and y coordinates, so replace y with x and let $x\to 0$ instead

$$\lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Not the same!!

Since different paths of approach give different values, $\boxed{\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist}}$

Here's what's going on graphically



Copy and paste into Mathematica to get interactive model

```
surface = Plot3D[x y/(x^2 + y^2), {x, -2, 2}, {y, -2, 2},
  Mesh -> 5,
  PlotStyle -> Directive[Orange, Opacity[0.8], Specularity[White, 30]]
  ];
xapproach = ParametricPlot3D[{t, 0, 0}, {t, -2, 2}, PlotStyle -> Directive[Blue, Thickness[.01]]];
yapproach = ParametricPlot3D[{0, t, 0}, {t, -2, 2}, PlotStyle -> Directive[Yellow, Thickness[.01]]];
diagonalapproach = ParametricPlot3D[{t, t, 1/2}, {t, -2 , 2}, PlotStyle -> Directive[Green, Thickness[01]]];

Show[{xapproach, yapproach, diagonalapproach, surface},
AxesLabel -> {x, y, z},
AxesOrigin -> {0, 0, 0},
Boxed -> False,
AxesStyle -> {Black, Black, Red},
RotationAction -> "Clip",
AspectRatio -> 1.5
]
```

Here's a faster way to do the same example

ex 1) (take 2) Compute $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Every* line through the origin is given by the equation $y=mx$ for various slopes m.
So replace y with mx and let $x\to 0$. Then at the end, see if your answer is constant for all m.

$$\lim_{x\to 0} \frac{x\cdot mx}{x^2+(mx)^2} = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

If $m=1$, $\frac{m}{1+m^2} = \frac{1}{2}$. If $m=2$, $\frac{m}{1+m^2} = \frac{2}{5}$

Since different values of m give different values of the limit, the limit $\boxed{\text{DNE}}$

* except the line $x=0$

Yet another way to do this problem is to use polar coordinates. In polar, every* linear path of approach to the origin is described by $r\to 0$. So converting to polar, taking the limit as $r\to 0$, and then considering the effect of various values of θ will handle every linear path of approach.

* I really mean it this time.

ex 1) (take 3) Compute $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Convert to polar and let $r\to 0$. $\lim_{r\to 0} \frac{r\sin\theta\, r\cos\theta}{(r\cos\theta)^2+(r\sin\theta)^2} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\cdot\sin\theta$

If $\theta=0$, $\cos\theta\cdot\sin\theta=0$. If $\theta=\pi/6$, $\cos\theta\cdot\sin\theta=\sqrt{3}/4$

Since different values of θ give different limit values, the limit $\boxed{\text{DNE}}$

ex 2) Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

Try to approach (0,0) along the line $y=mx$

$$\lim_{x\to 0} \frac{x\cdot(mx)^2}{x^2+(mx)^4} = \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0$$

{ Alert! This does <u>not</u> mean the original limit is zero!

Try to approach (0,0) along the curve $x=y^2$

$$\lim_{y\to 0} \frac{y^2\cdot y}{(y^2)^2+y^4} = \lim_{y\to 0} \frac{1}{y}$$

$\lim_{y\to 0^-} \frac{1}{y} = -\infty$ and $\lim_{y\to 0^+} \frac{1}{y} = +\infty$, so $\lim_{y\to 0} \frac{1}{y}$ does not exist.

Thus, $\boxed{\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} \text{ does not exist.}}$

Side note: Attempting this limit via polar coordinates is a bit subtle.

$$\lim_{r\to 0} \frac{r^3\cos\theta\sin^2\theta}{r^2\cos^2\theta+r^4\sin^4\theta}$$

$$\lim_{r\to 0} \frac{r\cos\theta\sin^2\theta}{\cos^2\theta+r^2\sin^4\theta}$$

if $\theta \neq \frac{\pi}{2}+k\pi$: $\lim_{r\to 0} \frac{0}{\cos^2\theta+0} = 0$

if $\theta = \frac{\pi}{2}+k\pi$: divide by zero error, try some other method

ex 3) Evaluate $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Try to approach (0,0) along the line $y=mx$

$$\lim_{x\to 0} \frac{3x^2\cdot mx}{x^2+(mx)^2} = \lim_{x\to 0} \frac{3mx^3}{(1+m^2)x^2} = \lim_{x\to 0} \frac{3mx}{1+m^2} = 0$$

{ Alert! This does <u>not</u> mean the original limit is zero!

Convert to polar and take the limit as $r\to 0$

$$\lim_{r\to 0} \frac{3(r\cos\theta)^2(r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2} = \lim_{r\to 0} \frac{3r^3\cos\theta\sin\theta}{r^2} = \lim_{r\to 0} 3r\cos\theta\sin\theta = 0$$

Use Squeeze Theorem

Since letting $r\to 0$ in polar accounts for all paths to (0,0) we may safely say

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0}$$

Tuesday September 22

Reminders

- WebAssign 11.2
- HW 5, all problems
- Work on graphing project

11.2 Limits (cont)

ex 4) Evaluate $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

This is really screaming out for polar coordinates

$\lim\limits_{r\to 0} r^2 \ln(r^2)$ (looks like "$0 \cdot -\infty$" so use l'Hôpital's rule)

$$\lim_{r\to 0} \frac{\ln(r^2)}{\frac{1}{r^2}} \xrightarrow{\text{l'Hôpital's}} \lim_{r\to 0} \frac{\frac{1}{r^2}\cdot 2r}{\frac{-2}{r^3}} = \lim_{r\to 0} \frac{2}{r}\cdot\frac{r^3}{-2} = 0$$

$$\boxed{\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = 0}$$

ex 5) Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4+4y^2}$

Try to approach $(0,0)$ along the line $y=mx$

$$\lim_{x\to 0} \frac{x^2(mx)e^{mx}}{x^4+4(mx)^2} = \lim_{x\to 0} \frac{m x^3 e^{mx}}{x^4+4m^2x^2} = \lim_{x\to 0} \frac{m x e^{mx}}{x^2+4m^2} = 0$$

{ Alert! This does <u>not</u> mean the original limit is zero!

Try to approach $(0,0)$ along the curve $y=x^2$

$$\lim_{x\to 0} \frac{x^2\cdot x^2\cdot e^{x^2}}{x^4+4x^4} = \lim_{x\to 0} \frac{e^{x^2}}{5} = \frac{1}{5}$$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4+4y^2} = \frac{1}{5}}$$

Continuity

A function $f(x,y)$ is continuous at (a,b) if $\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$

To prove $f(x,y)$ is continuous at (a,b),
(1) compute $\lim_{(x,y)\to(a,b)} f(x,y)$,
(2) compute $f(a,b)$, and
(3) say parts 1 and 2 are equal

ex 6) Is $g(x,y)$ continuous on its entire domain?

$$g(x,y) = \begin{cases} (x^2+y^2)\ln(x^2+y^2) & , \text{ if } (x,y)\neq(0,0) \\ 1 & , \text{ if } (x,y)=(0,0) \end{cases}$$

No, since $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = 0$, $f(0,0) = 1$, and $0 \neq 1$

ex 7) Is $g(x,y)$ continuous on its entire domain?

$$g(x,y) = \begin{cases} \dfrac{3xy^2}{x^2+y^2} & , \text{ if } (x,y)\neq(0,0) \\ 0 & , \text{ if } (x,y)=(0,0) \end{cases}$$

No, since $\lim_{(x,y)\to(0,0)} \dfrac{3xy^2}{x^2+y^2}$ does not exist.

Check-in 8

Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

Wednesday, September 22

Reminders
- Compile HW5 for André
- WebAssign 11.3
- Work on graphing project

11.3 Partial derivatives

(Look at GeoGebra demo at www.geogebra.com/m/kdphzd5k)

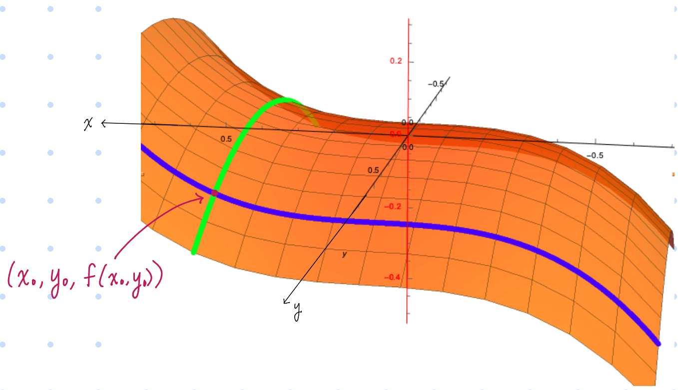
The partial derivatives of $f(x,y)$ with respect to x and y are, respectively,

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \qquad \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Other notation for the partial derivative: $f_x(x,y)$, f_x, $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial}{\partial x} f(x,y)$, $\frac{\partial}{\partial x}(f)$, $\frac{\partial z}{\partial x}$, $D_x f$
(and similarly for y)

ex 1) What is the sign of $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ for the function graphed below?

What is the sign of $\frac{\partial^2 f}{\partial x^2}(x_0,y_0)$ and $\frac{\partial^2 f}{\partial y^2}(x_0,y_0)$ for the function graphed below?



$\frac{\partial f}{\partial x}(x_0,y_0)$ is positive. $\frac{\partial f}{\partial y}(x_0,y_0)$ is negative.

$\frac{\partial^2 f}{\partial x^2}(x_0,y_0)$ is positive. $\frac{\partial^2 f}{\partial y^2}(x_0,y_0)$ is negative

We can compute $\frac{\partial f}{\partial x}$ by taking the derivative of $f(x,y)$ with respect to x as a variable and holding y constant.

ex 2) Let $f(x,y) = \frac{x-y}{x+y}$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}(0,-3)$

$$\frac{\partial f}{\partial x} = \frac{(x+y)(1)-(x-y)(1)}{(x+y)^2} = \boxed{\frac{2y}{(x+y)^2}}$$

$$\frac{\partial f}{\partial x}(0,-3) = \frac{2(-3)}{(0-3)^2} = \boxed{-\frac{2}{3}}$$

ex 3) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $\sin(xyz) = x + 2y + 3z$

We can't easily solve for z, so we must use implicit differentiation.

$$\frac{\partial}{\partial x}\left(\sin(xyz) = x + 2y + 3z\right)$$

$$\cos(xyz)\cdot y\left(x\cdot\frac{\partial z}{\partial x} + 1\cdot z\right) = 1 + 0 + 3\frac{\partial z}{\partial x}$$

$$xy\cos(xyz)\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial x} = 1 - yz\cos(xyz)$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3}}$$

We can also take higher-order derivatives.

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ (take derivative with respect to x twice)

$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ (take derivative with respect to y twice)

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ (take derivative with respect to x and then y)

$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ (take derivative with respect to y and then x)

Are these the same?

Clairaut's Theorem

Suppose f is defined on a disk D containing the point (a,b). If both f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$

Partial derivatives of parametrized surfaces.

[i.e. surfaces written as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ instead of $z = f(x,y)$]

For a parametrized surface $\vec{r}(u,v)$, its partial derivatives $\vec{r}_u$ and $\vec{r}_v$ are computed like this

$$\boxed{\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle , \quad \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle}$$

The geometric meaning of $\vec{r}_u$ is that it is a vector tangent to the surface $\vec{r}(u,v)$.
But even more than that, $\vec{r}_u$ is a vector tangent to the curves given by constant v.

ex 4) Let $\vec{r}(u,v) = \langle u\cos v, u \sin v, \sqrt{25-u^2} \rangle$, $0 \le u \le 5$ $\quad 0 \le v \le 2\pi$

(a) Draw $\vec{r}(u,v)$ and draw several curves given by constant u,v values.

Color on graph

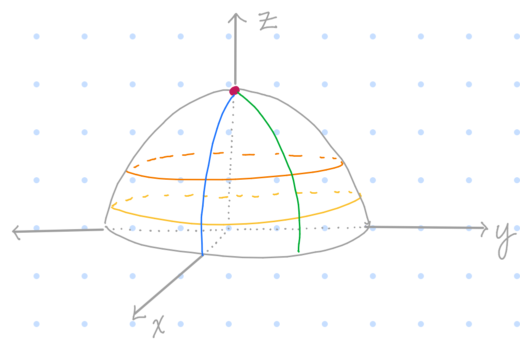
If $u=0$, $\vec{r}(0,v) = \langle 0, 0, 5 \rangle$

$u=3$, $\vec{r}(3,v) = \langle 3\cos\theta, 3\sin\theta, 4 \rangle$

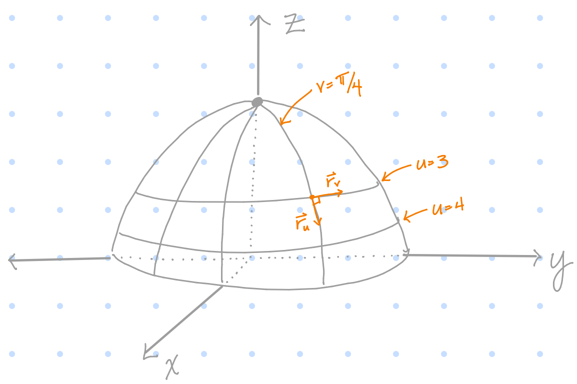
$u=4$, $\vec{r}(4,v) = \langle 4\cos\theta, 4\sin\theta, 3 \rangle$

If $v=0$, $\vec{r}(u,0) = \langle u, 0, \sqrt{25-u^2} \rangle$

$v=\pi/4$ $\vec{r}(u,\frac{\pi}{4}) = \langle \frac{\sqrt{2}}{2}u, \frac{\sqrt{2}}{2}u, \sqrt{25-u^2} \rangle$

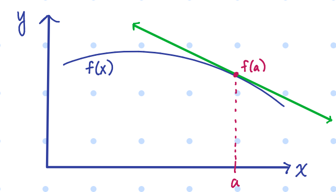


(b Draw $\vec{r}_u(3,\frac{\pi}{4})$ and $\vec{r}_v(3,\frac{\pi}{4})$ on the surface without doing any computations



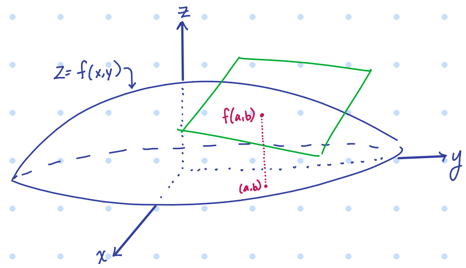
11.4 Tangent planes and linear approximation

Linear approximation of $y=f(x)$
(from Calc 1)



The tangent line of $f(x)$
at $x=a$ is a good approximation
for $f(x)$ near $x=a$

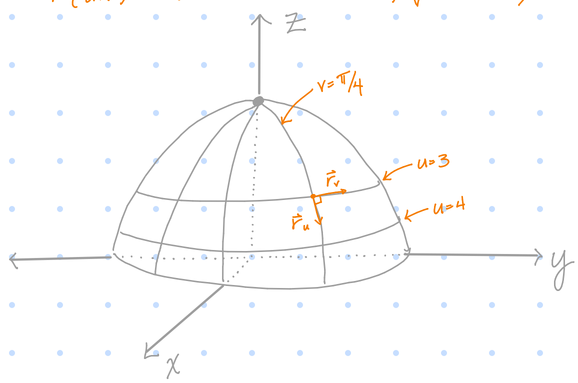
Linear approximation of $z=f(x,y)$
(from Calc 3)



The tangent plane of $f(x,y)$
at $(x,y)=(a,b)$ is a good approximation
for $f(x,y)$ near (a,b)

ex 1) Look at ex 4 in 11.3. Find the tangent plane to $\vec{r}(u,v)$ at the point $(3,\frac{\pi}{4})$

$\vec{r}(u,v)=\langle u\cos v,\ u\sin v,\ \sqrt{25-u^2}\ \rangle$



To find the equation of a tangent plane, we need a point on the plane and a vector normal to the plane.

Point : $\vec{r}(3,\frac{\pi}{4})=\langle \frac{3\sqrt{2}}{2},\ \frac{3\sqrt{2}}{2},\ 4\rangle$
Normal vector: $\langle 3,\ 3,\ 4\sqrt{2}\rangle$

Tangent plane:
$$\boxed{3\left(x-\frac{3\sqrt{2}}{2}\right)+3\left(y-\frac{3\sqrt{2}}{2}\right)+4\sqrt{2}(z-4)=0}$$

(computations to find normal vector)

$\vec{r}_u(u,v)=\langle \cos v\ ,\ \sin v\ ,\ -u(25-u^2)^{-1/2}\ \rangle$

$\vec{r}_v(u,v)=\langle -u\sin v,\ u\cos v,\ \ 0\ \rangle$

$\vec{r}_u(3,\frac{\pi}{4})=\langle \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ -\frac{3}{4}\rangle$

$\vec{r}_v(3,\frac{\pi}{4})=\langle -\frac{3\sqrt{2}}{2},\ \frac{3\sqrt{2}}{2},\ 0\rangle$

$\vec{r}_u\times\vec{r}_v=\langle \frac{9\sqrt{2}}{8},\ -\left(-\frac{9\sqrt{2}}{8}\right),\ 3\rangle$

convenient normal $\langle 3,3,4\sqrt{2}\rangle$

ex 2) Write the linearization of $\vec{r}(u,v)=\langle u\cos v,\ u\sin v,\ \sqrt{25-u^2}\rangle$ at $(3,\frac{\pi}{4})$.

Solve tangent plane equation for z.

$3\left(x-\frac{3\sqrt{2}}{2}\right)+3\left(y-\frac{3\sqrt{2}}{2}\right)+4\sqrt{2}(z-4)=0$

$3\left(x-\frac{3\sqrt{2}}{2}\right)+3\left(y-\frac{3\sqrt{2}}{2}\right)+4\sqrt{2}\,z-16\sqrt{2}=0$

$3\left(x-\frac{3\sqrt{2}}{2}\right)+3\left(y-\frac{3\sqrt{2}}{2}\right)-16\sqrt{2}=-4\sqrt{2}\,z$

$$\boxed{L(x,y)=-\frac{1}{4\sqrt{2}}\left(3\left(x-\frac{3\sqrt{2}}{2}\right)+3\left(y-\frac{3\sqrt{2}}{2}\right)-16\sqrt{2}\right)}$$

ex 3) Use a linearization of $f(x,y) = xe^{xy}$ at $(1,0)$ to approximate $f(1.1, -0.1)$

Step 1 option A: Find normal vector to tangent plane using parametrization

$\vec{r}(u,v) = \langle u, v, ue^{uv} \rangle$

"f_x"

$\vec{r}_u(u,v) = \langle 1, 0, e^{uv} + uve^{uv} \rangle$

$\vec{r}_v(u,v) = \langle 0, 1, u^2e^{uv} \rangle$ "f_y"

$\vec{r}_u \times \vec{r}_v = \langle -e^{uv}(1+uv), -u^2e^{uv}, 1 \rangle$

$\boxed{\vec{r}_u \times \vec{r}_v \Big|_{(1,0)} = \langle -1, -1, 1 \rangle}$

Step 1 option B: Find normal vector to tangent plane using formula for surfaces of the form $z = f(x,y)$

$f_x = e^{xy} + xye^{xy}$

$f_y = x^2e^{xy}$

$f_x \big|_{(1,0)} = 1$

$f_y \big|_{(1,0)} = 1$

normal vector formula: $\langle f_x, f_y, -1 \rangle$

$\boxed{\text{normal vector: } \langle 1, 1, -1 \rangle}$

Step 2: Find point of tangency

$x = 1$

$y = 0$

$z = f(1,0) = 1$

$\boxed{\text{point: } (1, 0, 1)}$

Step 3: Write linearization

$L(x,y) = 1 + 1(x-1) + 1(y-0)$

$\boxed{L(x,y) = x + y}$

Step 4: Use linearization by plugging in $(1.1, -0.1)$

$L(1,0) = 1.1 + (-0.1) = \boxed{1}$

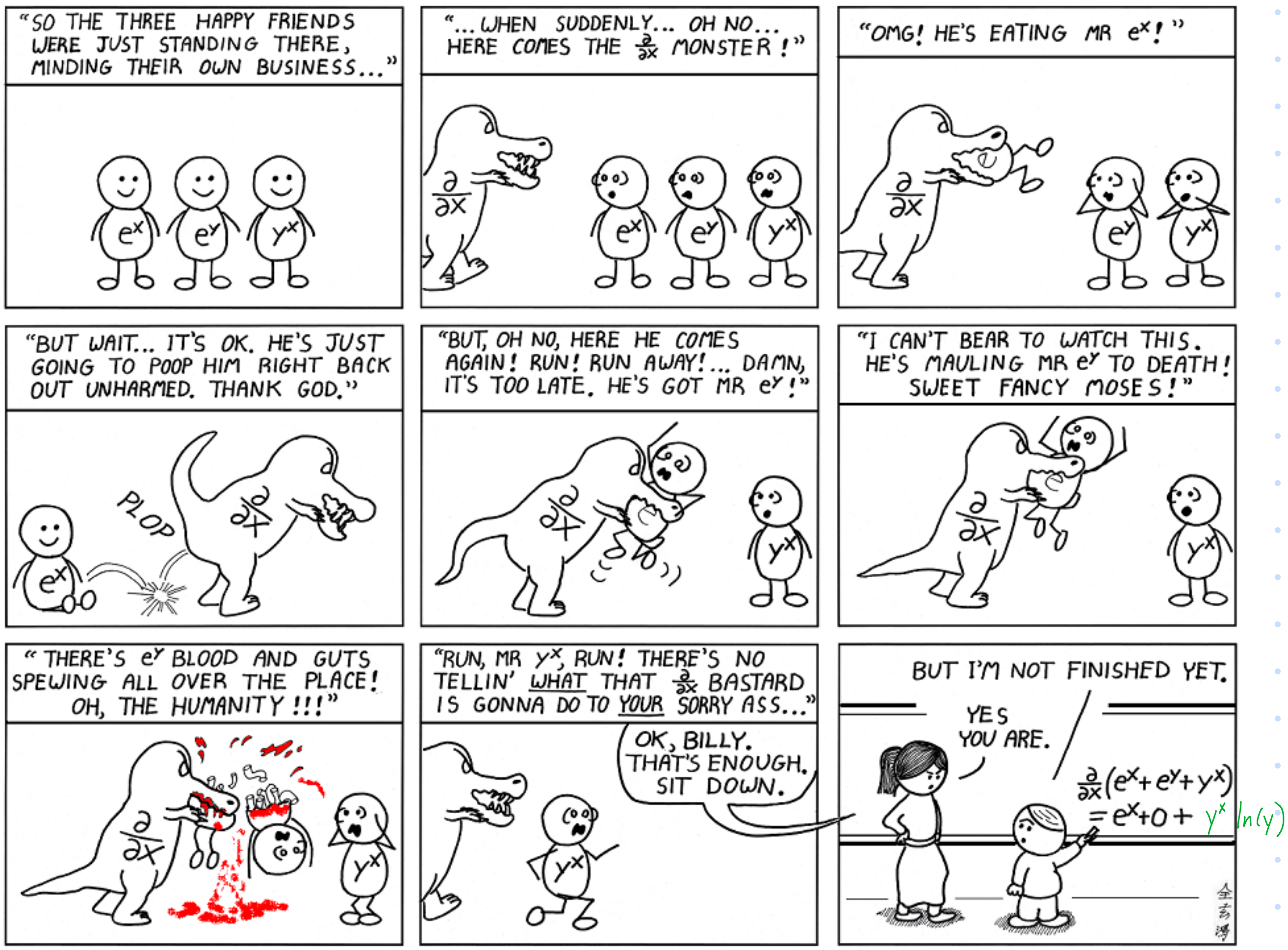
Step 5: State conclusion

For $f(x,y) = xe^{xy}$, we know $\boxed{f(1.1, -0.1) \approx 1}$

Monday, September 27

Reminders

- Study for Quiz 3. Review problems on Piazza
- WebAssign 11.5
- HW 6, all except A2
- HW 7, section 11.5, A1, A2



11.5 Chain rule

In calc 1, we used the chain rule to compute the derivative of a composition of functions.

Chain rule: Given $f(x)$ and $g(x)$, $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$ ...or $\frac{df}{dg} \cdot \frac{dg}{dx}$

In calc 3, function composition itself is more complicated. The output of an inner function must be compatible with the domain of the outer function.

Consider the following functions:

$\mathbb{R} \to \mathbb{R}$	$h(x) = e^x$
$\mathbb{R}^2 \to \mathbb{R}$	$f(x, y) = x^2 + y^2$
$\mathbb{R}^2 \to \mathbb{R}^3$	$\mathbf{g}(x, y) = \langle x + y, 3x - y, 2x + y \rangle$
$\mathbb{R} \to \mathbb{R}^3$	$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$
$\mathbb{R} \to \mathbb{R}^2$	$\mathbf{p}(t) = \langle -\sin t, \cos t \rangle$
$\mathbb{R}^3 \to \mathbb{R}^2$	$\mathbf{w}(x, y, z) = \langle 2x, 2y \rangle$

For each of the compositions below, indicate whether or not they are defined by circling (a) or (b). If they are defined, fill in the boxes to show the dimensions of the input and output spaces.

1. (a) $f \circ h : \mathbb{R}^{\square} \mapsto \mathbb{R}^{\square}$
   (b) $f \circ h$ is not defined.
2. (a) $h \circ f : \mathbb{R}^{\boxed{2}} \mapsto \mathbb{R}^{\boxed{1}}$
   (b) $h \circ f$ is not defined.
3. (a) $\mathbf{g} \circ \mathbf{w} : \mathbb{R}^{\boxed{3}} \mapsto \mathbb{R}^{\boxed{3}}$
   (b) $\mathbf{g} \circ \mathbf{w}$ is not defined.
4. (a) $\mathbf{w} \circ \mathbf{g} : \mathbb{R}^{\boxed{2}} \mapsto \mathbb{R}^{\boxed{2}}$
   (b) $\mathbf{w} \circ \mathbf{g}$ is not defined.
5. (a) $f \circ \mathbf{g} : \mathbb{R}^{\square} \mapsto \mathbb{R}^{\square}$
   (b) $f \circ \mathbf{g}$ is not defined.
6. (a) $\mathbf{g} \circ \mathbf{p} : \mathbb{R}^{\boxed{1}} \mapsto \mathbb{R}^{\boxed{3}}$
   (b) $\mathbf{g} \circ \mathbf{p}$ is not defined.
7. (a) $\mathbf{r} \circ \mathbf{w} : \mathbb{R}^{\square} \mapsto \mathbb{R}^{\square}$
   (b) $\mathbf{r} \circ \mathbf{w}$ is not defined.
8. (a) $f \circ \mathbf{p} : \mathbb{R}^{\boxed{1}} \mapsto \mathbb{R}^{\boxed{1}}$
   (b) $f \circ \mathbf{p}$ is not defined.
9. There are 6 more meaningful compositions of the given functions that were not named above. List at least three of them.

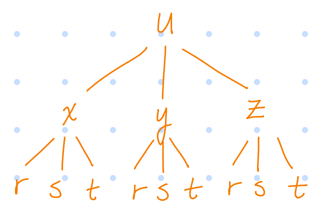
Chain rule (general version)

Suppose u is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$. Then for each $i = 1, 2, \dots, m$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

ex 1) If $u = x^4 y + y^2 z^2$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$, find $\frac{\partial u}{\partial s}$ when $r=2$, $s=1$, $t=0$

Step 1: tree diagram



Step 2: Use tree to write chain rule for $\partial u/\partial s$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

... or $u_s = u_x x_s + u_y y_s + u_z z_s$

Step 3: Compute derivatives.

$$\frac{\partial u}{\partial s} = (4x^3 y)(re^t) + (x^4 + 2yz^2)(2rse^{-t}) + (2y^2 z)(r^2 \sin t)$$

Step 4: Plug in values for r, s, t. [Notice! The given values for r, s, t also determine values for x, y, z]

$x(2,1,0) = 2$
$y(2,1,0) = 2$
$z(2,1,0) = 0$

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{r=2\\ s=1\\ t=0}} = (4(2)^3 2)(2e^0) + (2^4 + 2\cdot 2\cdot 0^2)(2\cdot 2\cdot 1\cdot e^0) + (2\cdot 2^2\cdot 0)(2^2 \sin 0)$$

$$= 128 + 64 + 0$$

$$= \boxed{192}$$

[example 2 postponed to tomorrow]

ex 3)

34. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of $0.15°$ C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$.

a. What is the significance of the signs of these partial derivatives?

b. Estimate the current rate of change of wheat production, dW/dt.

Information given

$$W = W(T, R)$$

$$\frac{dT}{dt} = 0.15 \qquad \frac{dR}{dt} = -0.1$$

$$\frac{\partial W}{\partial T} = -2 \qquad \frac{\partial W}{\partial R} = 8$$

(a) $\left[\frac{\partial W}{\partial T} = -2\right]$ At current production levels, each 1°C increase in temperature will reduce wheat production by about 2 units

$\left[\frac{\partial W}{\partial R} = 8\right]$ At current production levels, each 1 cm increase in rainfall will increase wheat production by about 8 units

(b) $$\frac{dW}{dt} = \frac{\partial W}{\partial T}\frac{dT}{dt} + \frac{\partial W}{\partial R}\frac{dR}{dt}$$

$$= (-2)(0.15) + (8)(-0.1)$$

$= -1.1$ units of wheat per year.

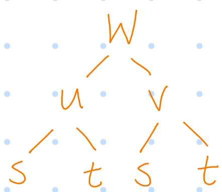
Tuesday September 29

Reminders

- ! Important ! Do Quiz 3 between 7 - 10 pm

11.5 Chain rule (cont)

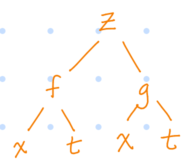
Warm-up: Let $W(s,t) = F(u(s,t), v(s,t))$. Draw a tree diagram and write the chain rule for W_s.



$$W_s = W_u u_s + W_v v_s$$

ex 2) [HW 7 prob A1] Show that any function of the form $z = f(x+at) + g(x-at)$ is a solution to the wave equation $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$. [a is a non-negative constant]

Step 1: tree diagram



Step 2: Use tree to write chain rule for $\frac{\partial^2 z}{\partial t^2}$, $\frac{\partial^2 z}{\partial x^2}$

Try on your own

$z_t = z_f f_t + z_g g_t$

$z_x = \ldots$

$z_{tt} = (z_t)_t = (z_f)_t f_t + z_f (f_t)_t + (z_g)_t g_t + z_g (g_t)_t$

$z_{xx} = \ldots$

$= (z_{ff} f_t + z_{fg} g_t) f_t + z_f f_{tt} + (z_{gf} f_t + z_{gg} g_t) g_t + z_g g_{tt}$

$= z_{ff} (f_t)^2 + 2 z_{fg} f_t g_t + z_{gg} (g_t)^2 + z_f f_{tt} + z_g g_{tt}$

Step 3: Compute derivatives

$z_{tt} = 0 + 0 + 0 + 1 \cdot a^2 f''(x+at) + 1 \cdot a^2 g''(x-at)$

$= a^2 \left(f''(x+at) + g''(x-at) \right)$

Step 4: Compute right-hand side and ~~hope~~ show the two sides are equal.

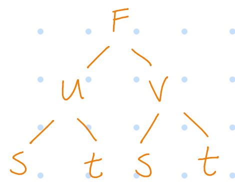
11.5 Prob 14

14. Let $W(s,t) = F(u(s,t), v(s,t))$, where F, u, and v are differentiable, and

$$\begin{array}{ll} u(1,0) = 2 & v(1,0) = 3 \\ u_s(1,0) = -2 & v_s(1,0) = 5 \\ u_t(1,0) = 6 & v_t(1,0) = 4 \\ F_u(2,3) = -1 & F_v(2,3) = 10 \end{array}$$

Find $W_s(1,0)$ and $W_t(1,0)$.

* Popular exam question



$$W_s = F_u U_s + F_v V_s$$

$$W_t = F_u U_t + F_v V_t$$

$$W_s(1,0) = F_u(u(1,0), v(1,0))\, U_s(1,0) + F_v(u(1,0), v(1,0))\, V_s(1,0)$$

$$= F_u(2, 3)(-2) + F_v(2, 3)(5)$$

$$= (-1)(-2) + (10)(5)$$

$$= \boxed{52}$$

$$W_t(1,0) = F_u(u(1,0), v(1,0))\, u_t(1,0) + F_v(u(1,0), v(1,0))\, V_t(1,0)$$

$$= (-1)(6) + (10)(4)$$

$$= \boxed{34}$$

<u>Note about WebAssign</u>

Implicit Function Theorem (see book) is mentioned in last 3 questions of 11.5 WebAssign.
It's not important for us - either follow the given formula or do implicit differentiation the same way you have before

Wednesday September 29

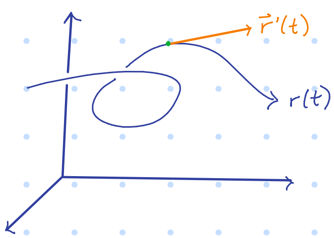
Reminders

- Compile HW 6 for André
- Submit Quiz 3 corrections by 10 pm

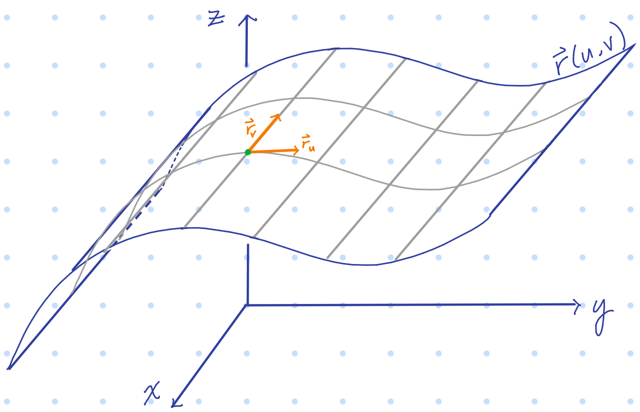
11.6 Directional derivatives and the gradient vector

Derivatives we know (and love) so far:

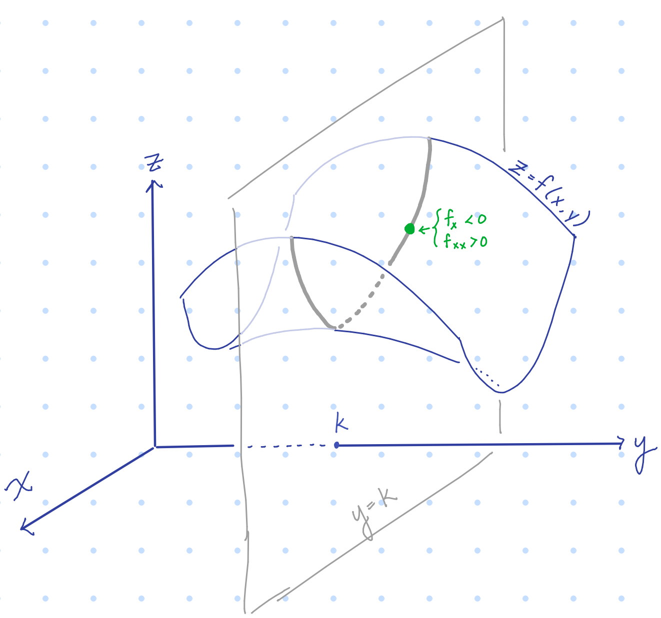
- For vector-valued, one-parameter curves $\vec{r}(t)$, we know $\vec{r}'(t)$ is the tangent vector to the curve.



- For vector-valued, two-parameter surfaces $\vec{r}(u,v)$, we know the partial derivatives $\vec{r}_u$ and $\vec{r}_v$ are vectors tangent to the surface (i.e. they lie in the tangent plane).



- For scalar-valued functions $z = f(x,y)$, the partial derivatives f_x and f_y are scalars that give the slope of the curve given by fixing a constant value for y and x, respectively.



Now introducing ... the gradient!

- For a scalar valued function $z = f(x,y)$ [or $w = f(x,y,z)$] the gradient of f is the vector of partial derivatives $\langle f_x, f_y \rangle$ [or $\langle f_x, f_y, f_z \rangle$]. In formal notation,

$$\text{grad } f = \nabla f = \langle f_x, f_y \rangle \quad [\text{or } \langle f_x, f_y, f_z \rangle]$$

ex 1) Compute the gradient of $f(x,y) = 9 - x^2 - y^2$. Draw the level curves and some gradient vectors on the same axes.

$\nabla f = \langle -2x, -2y \rangle$

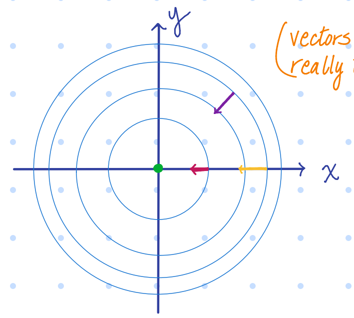
Some specific values of ∇f

$\nabla f(0,0) = \langle 0, 0 \rangle$

$\nabla f(1,0) = \langle -2, 0 \rangle$

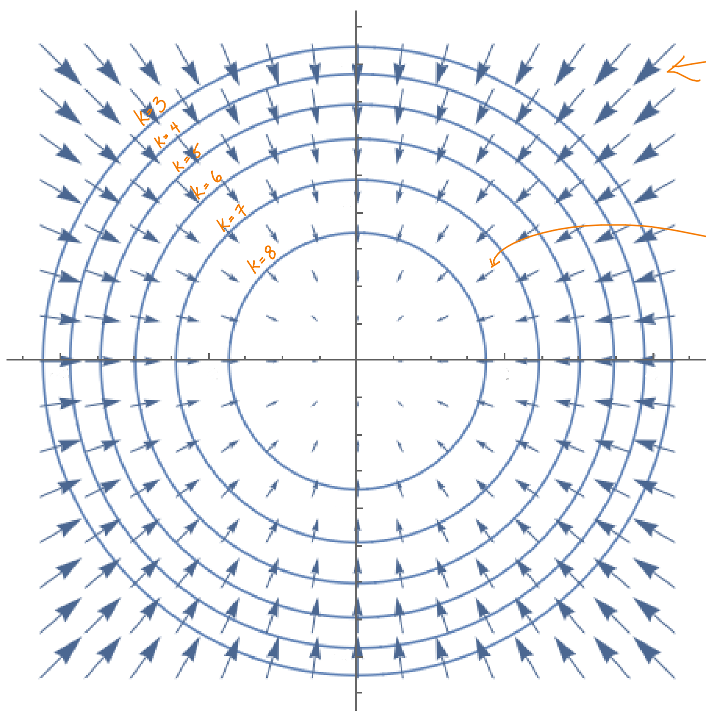
$\nabla f(2,0) = \langle -4, 0 \rangle$

$\nabla f(\sqrt{2}, \sqrt{2}) = \langle -2\sqrt{2}, -2\sqrt{2} \rangle$



(vectors not really to scale)

Better picture by Mathematica



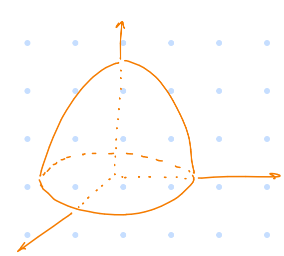
Big arrow means big rate of change in that direction, i.e. very steep

Smaller arrow means smaller growth i.e. less steep.

Notice every arrow points "up" toward higher ground.

Notice every vector is orthogonal to the circles of the contour plot.

This is consistent with our knowledge that $z = 9 - x^2 - y^2$ is a downward paraboloid.



Properties of the gradient

- Its graph is drawn in the domain of f. [ex) For $f(x,y)$, draw ∇f on $\mathbb{R}^2$]
- Its graph is a bunch of vectors
- Each gradient vector points in the direction of fastest increase (i.e. steepest uphill)
- Each gradient vector at a point is orthogonal to the level curve through that point
- The magnitude of each gradient vector is the rate of change of f in the direction of the vector.

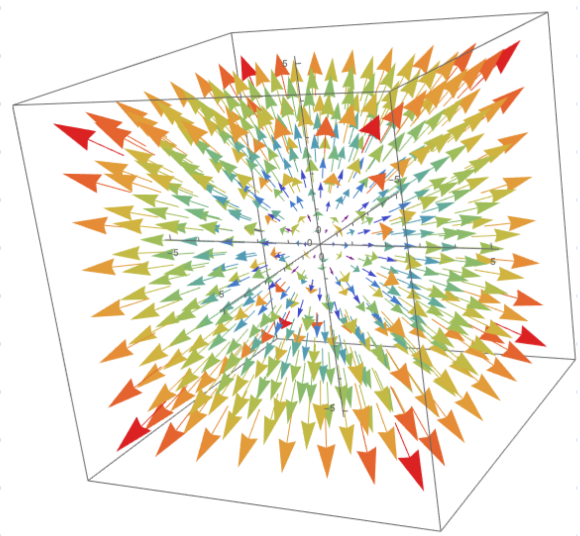
ex 2) [Now with more dimensions!] Let $T(x,y,z) = x^2 + y^2 + z^2$ be the temperature of a room.

(a) Compute ∇T and use Mathematica to plot ∇T.

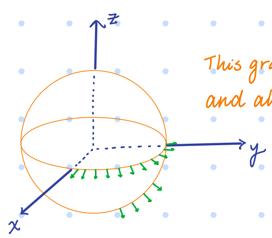
$\nabla T = \langle 2x, 2y, 2z \rangle$

Mathematica code:

```
VectorPlot3D [{2x, 2y, 2z}, {x, -4, 4}, {y, -4, 4}, {z, -4,4}, VectorColorFunction -> "Rainbow"]
```



(b) Draw a few level surfaces to convince yourself that the gradient vectors are orthogonal to level surfaces.



This gradient points "outward" and all level surfaces are spheres.

[Important thought for later: If a surface can be interpreted as the level surface of some higher-dimensional function f, then ∇f produces vectors orthogonal to the surface. This makes finding tangent planes very easy.]

(b) A sweaty bee is at $(-3,0,4)$. In what direction should it fly to cool down fastest?

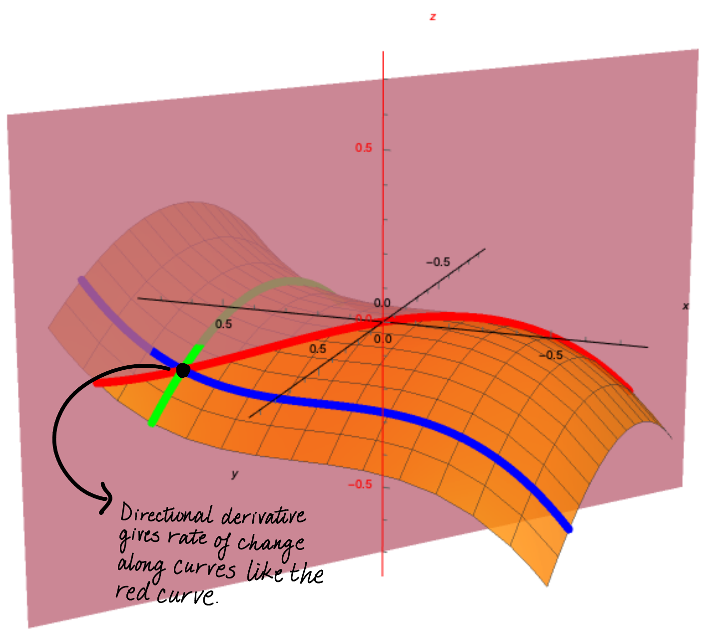
$\nabla T(-3,0,4) = \langle -6, 0, 8 \rangle$. The direction of fastest temperature decrease is $\langle 6, 0, -8 \rangle$.

Here's yet another derivative!

- For a scalar-valued function f and a fixed <u>unit</u> vector $\vec{u}$, the directional derivative of f in the direction of $\vec{u}$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$D_{\vec{u}} f$ is a scalar that gives the rate of change of f in the direction of $\vec{u}$. It is the same idea as f_x or f_y, but along an arbitrary vector $\vec{u}$ instead of along curves given by fixed y or x.



ex 3) If the sweaty bee in example 2 flies in the direction $\langle 1, 1, 0 \rangle$, is it getting cooler?

$\nabla T = \langle 2x, 2y, 2z \rangle$

$\nabla T(-3, 0, 4) = \langle -6, 0, 8 \rangle$

unit vector of $\langle 1, 1, 0 \rangle$ is $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$

$D_{\vec{u}} T = \langle -6, 0, 8 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$

$= -3\sqrt{2}$ Yes the bee is cooling off, but not as quickly as it would BEE* if it flew in the direction $\langle 6, 0, -8 \rangle$, opposite the gradient.

Friday, October 2

Reminders

- All Week 6 WebAssign due Sunday before midnight
- HW 7, section 11.6
- Work on graphing project (due Oct 6, Tues!)
- Review Calc 1 concept of max/min problems.

Definition. The *directional derivative* of f at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Part 1: Find an alternate definition for the directional derivative of f at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$.

Define a new function $g : \mathbb{R} \to \mathbb{R}$ as follows

$$g(h) = f(x_0 + ha, y_0 + hb).$$

1. Verify that $g'(0) = D_{\mathbf{u}} f(x_0, y_0)$ using the limit definitions of the derivative and the directional derivative.

2. Use the chain rule and the fact that $g(h) = f(x(h), y(h))$ where $x(h) = x_0 + ha$ and $y(h) = y_0 + hb$ to find $g'(0)$.

3. Use your answers to parts (1) and (2) to give a formula for $D_{\mathbf{u}} f(x_0, y_0)$.

4. What does the directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ represent?

Part 2: Cliff is hiking Montaña Derivado. Go to `https://www.geogebra.org/m/raxnv2b8` to see the surface $f(x, y)$ that represents the mountain.

1. Each of the following points represents a location on the mountain the Cliff has hiked by. For each point, find the directional derivative in the direction of $\mathbf{u}$.

   (i) $(x_0, y_0) = (-1, -1), \mathbf{u} = \langle 0.9, -0.43\rangle$ $\qquad D_{\mathbf{u}}f(x_0, y_0) =$ ________

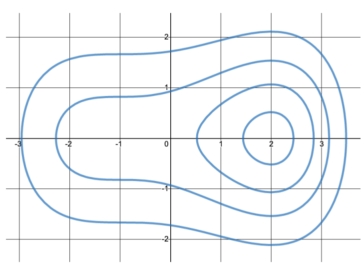
   (ii) $(x_0, y_0) = (2, 1), \mathbf{u} = \langle -0.8, -0.6\rangle$ $\qquad D_{\mathbf{u}}f(x_0, y_0) =$ ________

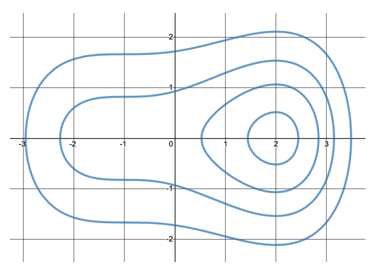
   (iii) $(x_0, y_0) = (1, 0), \mathbf{u} = \langle 0, 1\rangle$ $\qquad D_{\mathbf{u}}f(x_0, y_0) =$ ________

2. What does the values of $D_{\mathbf{u}}f(x_0, y_0)$ you found above represent?

3. For each of the following points, find the unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(x_0, y_0)$ is the largest and the unit vector $\mathbf{v}$ such that $D_{\mathbf{v}}f(x_0, y_0)$ is zero. Draw the vectors on the contour plots below.

	(x_0, y_0)	$D_{\mathbf{u}}f(x_0, y_0)$ is the largest	$D_{\mathbf{v}}f(x_0, y_0)$ is zero
(i)	$(1.5, 1.5)$	$\mathbf{u} =$ ________	$\mathbf{v} =$ ________
(ii)	$(3, -1)$	$\mathbf{u} =$ ________	$\mathbf{v} =$ ________
(iii)	$(-1, -.8)$	$\mathbf{u} =$ ________	$\mathbf{v} =$ ________





4. Fill in the blanks.

   (a) The directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is *largest* when $\mathbf{u}$ is ______________________________ to the contour line at (x_0, y_0). In this case, $\mathbf{u}$ is pointing in the direction where the mountain is ______________________________.

   (b) The directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is *zero* when $\mathbf{u}$ is ______________________________ to the contour line at (x_0, y_0). In this case, $\mathbf{u}$ is pointing in the direction where the mountain is ______________________________.

Definition. If f is a function of two variables x and y, then the *gradient* of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

Alternate definitions. The *directional derivative* of f at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$ can also be written as

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= \underline{\hspace{4cm}} \cdot \mathbf{u} \\ &= \underline{\hspace{4cm}} \cos(\theta) \end{aligned}$$

where θ is the angle between ____________________ and $\mathbf{u}$.

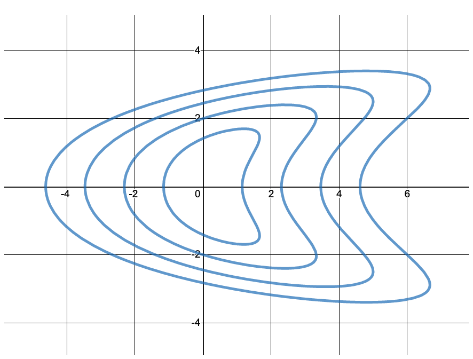
5. How can we use these alternate definitions to prove the statements in Problem 4?

Part 3: Now, Cliff is hiking Montagne Vecteur. The surface of the mountain is given by the following equation

$$f(x,y) = 4 - 0.5\sqrt{3x^2 - 2.5xy^2 + y^4}.$$

1. Find $\nabla f(x,y)$.

2. Go to `https://www.geogebra.org/m/fcvck9ca` to see the surface $f(x,y)$. A contour plot of the mountain is shown below. Find the following gradient vectors and plot each on the contour plot below.



(i) $\nabla f(-1.3, 1.42) =$ __________

(ii) $\nabla f(0.55, -1.54) =$ __________

(iii) $\nabla f(4.01, 1.15) =$ __________

(iv) $\nabla f(0,0) =$ __________

3. Cliff is at the point $(-1, -1)$. He is debating whether he should take the steepest path to the top of the mountain or stay at his current elevation and go around.

   (a) What direction should Cliff head if he wants to take the steepest path to the top of the mountain? Explain.

   (b) What direction should Cliff head if he wants to go around the mountain, staying at his current elevation? Explain.

4. After deciding to go around, Cliff is at the point $(2, 1.27)$ when he spots a bear at point $(1.6, 1.1)$.

   (a) What direction should Cliff run if he wants to run in the opposite direction of the bear? Explain.

   (b) What direction should Cliff run if he wants to run down the steepest part of the mountain? Explain.

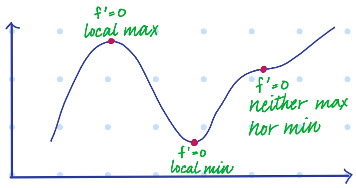
Monday, October 5

Reminders

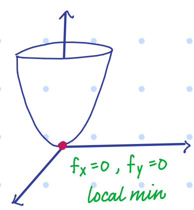
- Graphing project due tomorrow (before midnight)
  - upload reflection + pledge as check-in 11
  - email me the .NB file
- WebAssign 11.7, problems 2,3,4,5

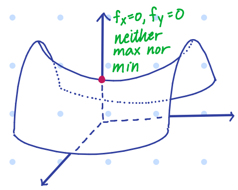
11.7 Max + min values

In Calc 1



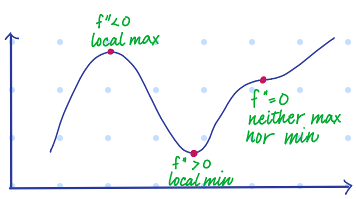
In Calc 3



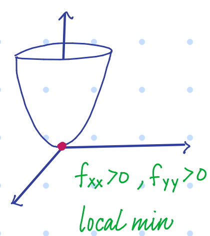


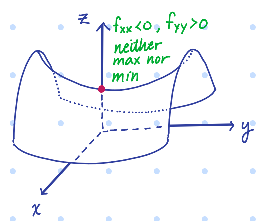
Takeaway: Finding points where $f_x = 0$ and $f_y = 0$ (in other words, $\nabla f = \vec{0}$) will identify possible candidates for local max/min of $f(x,y)$.

In Calc 1



In Calc 3





Takeaway: If the second derivative exists, it can help detect max/min points.

The analogue of the single-variable $f''(x)$ from Calc 1 is a 2×2 matrix of partial derivatives called the Hessian (name not important).

Hessian of $f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ Determinant of Hessian $= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

Second derivative test: Suppose the second partial derivatives of $f(x,y)$ are continuous on a disk with center (a,b) and $f_x(a,b)=0$, $f_y(a,b)=0$. Let $D = D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - [f_{xy}(a,b)]^2$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum
(b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum
(c) If $D<0$, then $f(a,b)$ is a saddle point
(d) If $D=0$, then this test gives no conclusion. Time to get creative!

ex1) Classify all critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$

Step 1: Find critical points by solving $f_x = 0$, $f_y = 0$

$f_x = 3x^2 - 3y$
$f_y = 3y^2 - 3x$

$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases}$

$\begin{cases} x^2 = y \\ y^2 = x \end{cases}$

$y^4 = y$
$y^4 - y = 0$
$y(y^3 - 1) = 0$
$y = 0$ $y = 1$ (protip: complete each ordered pair immediately)
$x = 0$ $x = \pm 1$ ← checking both equations shows $(-1,1)$ doesn't work

Critical points: $\boxed{(0,0) \quad (1,1)}$

Step 2: Compute D

$f_{xx} = 6x$
$f_{yy} = 6y$
$f_{xy} = -3$

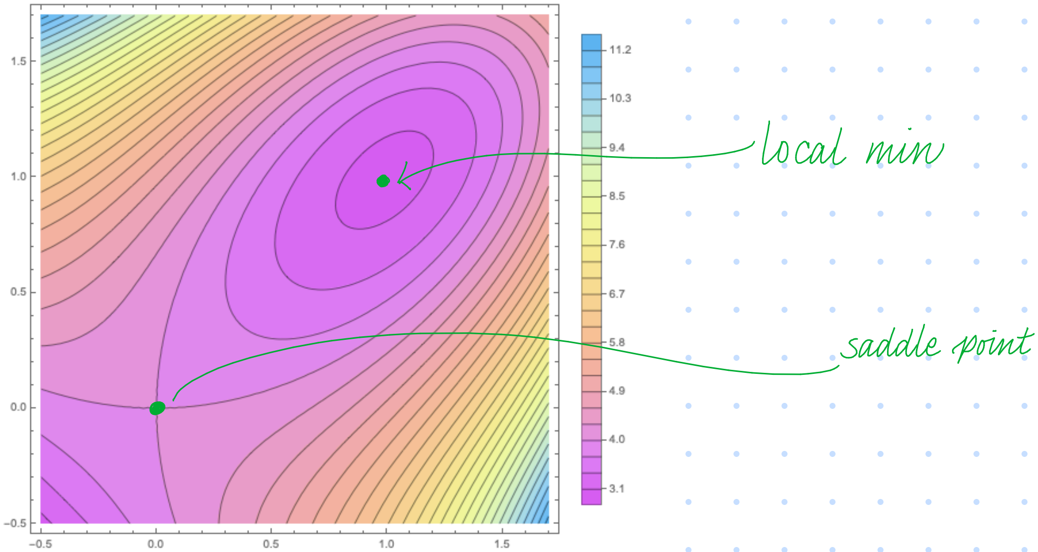
$D = f_{xx}f_{yy} - [f_{xy}]^2$

$\boxed{D = 36xy - 9}$

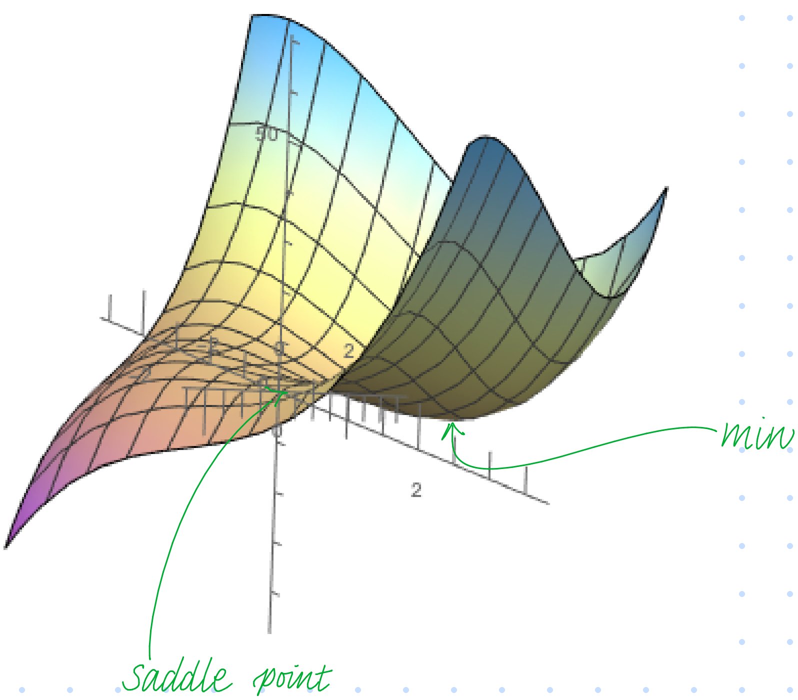
Step 3: Analyze each critical point

$D(0,0) = -9$, so $\boxed{f \text{ has a saddle point at } (0,0)}$
$D(1,1) = 27$ and $f_{xx}(1,1) = 6$, so $\boxed{f \text{ has a local min at } (1,1)}$

ex 2) Classify all critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$ by analyzing the contour plot



Here's what the surface looks like.



ex 3) [section 11.7 prob 13] Find local extrema and saddle points of $f(x,y) = (x^2+y^2)e^{y^2-x^2}$

Step 1: Find critical points

$$f_x = 2xe^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(-2x)$$
$$= 2xe^{y^2-x^2}(1-x^2-y^2)$$
$$f_y = 2ye^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(2y)$$
$$= 2ye^{y^2-x^2}(1+x^2+y^2)$$

$$\begin{cases} 2xe^{y^2-x^2}(1-x^2-y^2) = 0 \\ 2ye^{y^2-x^2}(1+x^2+y^2) = 0 \end{cases}$$

- If $x=0$, then $2ye^{y^2}(1+y^2)=0$, which is only true when $y=0$

$\boxed{\text{critical point } (0,0)}$

- $e^{y^2-x^2}$ is never zero
- If $1-x^2-y^2=0$, then $x^2+y^2=1$. Plug in to second equation to get $2ye^{y^2-x^2}(1+1)=0$, which is only true when $y=0$. This means $x=\pm 1$

$\boxed{\text{critical points } (1,0),\ (-1,0)}$

Step 2: Compute D and analyze critical points

$$f_{xx} = 2e^{y^2-x^2}(1-x^2-y^2) + 2x\left[e^{y^2-x^2}(-2x)(1-x^2-y^2) + e^{y^2-x^2}(-2x)\right]$$
$$= 2e^{y^2-x^2}(1-x^2-y^2) - 4x^2e^{y^2-x^2}(1-x^2-y^2) - 4x^2e^{y^2-x^2}$$
$$= 2e^{y^2-x^2}\left[(1-x^2-y^2) - 2x^2(1-x^2-y^2) - 2x^2\right]$$

$$f_{yy} = 2e^{y^2-x^2}(1+x^2+y^2) + 2y\left[e^{y^2-x^2}(2y)(1+x^2+y^2) + e^{y^2-x^2}(2y)\right]$$
$$= 2e^{y^2-x^2}\left[(1+x^2+y^2) + 2y^2(1+x^2+y^2) + 2y^2\right]$$

$$f_{xy} = 2xe^{y^2-x^2}(2y)(1-x^2-y^2) + 2xe^{y^2-x^2}(-2y)$$
$$= 4xye^{y^2-x^2}(-x^2-y^2)$$

$f_{xx}(0,0) = 2[1] = 2$
$f_{yy}(0,0) = 2[1] = 2$
$f_{xy}(0,0) = 0$

$D(0,0) = 2\cdot 2 - 0^2 = 4$

$\boxed{f \text{ has a local min of } 0 \text{ at } (0,0)}$

$f_{xx}(1,0) = 2e^{-1}[0-0-2] = -4/e$
$f_{yy}(1,0) = 2e^{-1}[2] = 4/e$

$\boxed{f \text{ has a saddle point at } (1,0)}$

$f_{xx}(-1,0) = 2e^{-1}[-2] = -4/e$
$f_{yy}(-1,0) = 2e^{-1}[2] = 4/e$

$\boxed{f \text{ has a saddle point at } (1,0)}$

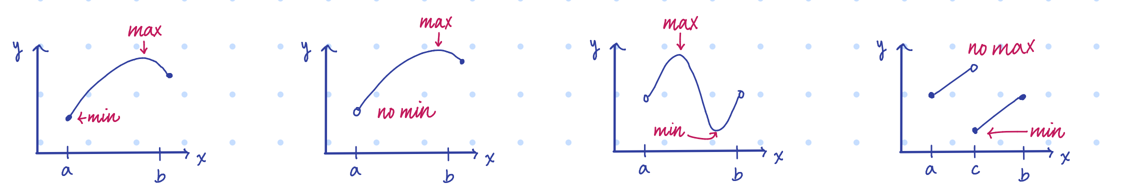
Tuesday, October 6

Reminders

- Graphing project due before midnight tonight
  - Upload .PDF of reflections + pledge as Check-in 11
  - Email .NB file to cherry.ng@colorado.edu
- Ask HW7 questions on OneNote, if desired
- Finish WebAssign 11.7

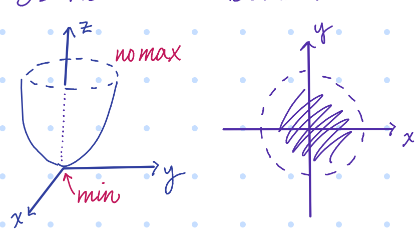
> **Extreme Value Theorem**
>
> If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f is guaranteed to have an absolute max and absolute min on D

In Calc 1



continuous on $[a,b]$ — continuous on $(a,b]$ — continuous on (a,b) — continuous on $[a,c) \cup [c,b]$

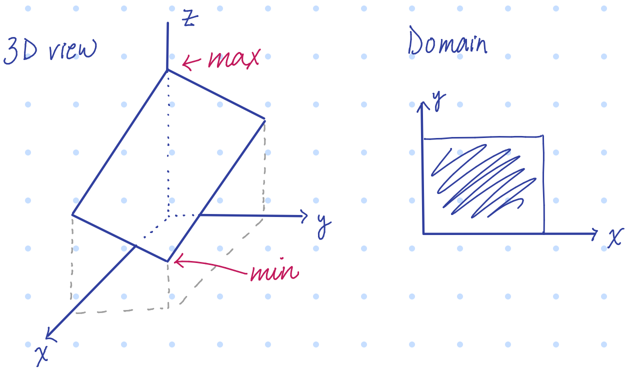
In Calc 3



$z = x^2+y^2$ — $x^2+y^2 < 1$

✓ f is continuous?
✓ domain is bounded?
✗ domain is closed?

$\Longrightarrow$ existence of absolute extrema not guaranteed



✓ f is continuous?
✓ domain is bounded?
✓ domain is closed?

$\Longrightarrow$ existence of absolute extrema is guaranteed!

Procedure for finding absolute extrema

1.) Find value of f at all critical points of f in domain (note: do not use $D = f_{xx}f_{yy} - [f_{xy}]^2$)

2.) Find extreme values of f on boundary of domain (note: This part is a Calc 1-style max/min problem)

3.) Compare values from steps 1 and 2 to find the biggest/smallest.

ex 4) [section 11.7 problem 29] Find the absolute extrema of $f(x,y) = x^2 + y^2 + x^2y + 4$ on the set $D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$

Step 1: Find critical points of f in D

$f_x = 2x + 2xy$
$f_y = 2y + x^2$

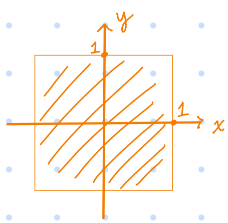
$\begin{cases} 2x(1+y) = 0 \\ 2y + x^2 = 0 \end{cases}$

- If $x=0$, $y=0$ — $\boxed{\text{crit pt } (0,0)}$
- If $1+y=0$, $y=-1$, and $-2+x^2=0$, so $x=\pm\sqrt{2}$. But $(\pm\sqrt{2},-1)$ is not in D. — no crit pts
- If $2y+x^2=0$, $y=\frac{-x^2}{2}$, so plug into equation 1 to get $2x\left(1-\frac{x^2}{2}\right)=0$, which gives $x=0$, $x=\pm\sqrt{2}$. — no crit pts

These are redundant

Step 2: Find extrema on boundary by using Calc 1 skills.

Picture of D



$-1 \le x \le 1$
$-1 \le y \le 1$

Boundary of D

$\vec{r}_1(t) = \langle 1, t \rangle$, $-1 \le t \le 1$
$\vec{r}_2(t) = \langle -1, t \rangle$, $-1 \le t \le 1$
$\vec{r}_3(t) = \langle t, 1 \rangle$, $-1 \le t \le 1$
$\vec{r}_4(t) = \langle t, -1 \rangle$, $-1 \le t \le 1$

Finding extrema on boundary

$f(\vec{r}_1(t)) = 1^2 + t^2 + t + 4$
$= t^2 + t + 5$
$\frac{d}{dt} f(\vec{r}_1(t)) = 2t + 1$
$0 = 2t + 1$
$t = -\frac{1}{2}$

$\boxed{\begin{array}{l}\text{crit pt } (1,-\frac{1}{2}) \\ \text{endpoints } (1,-1), (1,1)\end{array}}$

$f(\vec{r}_2(t)) = 1 + t^2 + t + 4$
$\frac{d}{dt} f(\vec{r}_2(t)) = 2t + 1$
$0 = 2t + 1$
$t = -\frac{1}{2}$

$\boxed{\begin{array}{l}\text{crit pt } (-1,-\frac{1}{2}) \\ \text{endpoints } (-1,1), (-1,-1)\end{array}}$

$f(\vec{r}_3(t)) = t^2 + 1 + t^2 + 4$
$= 2t^2 + 5$
$\frac{d}{dt} f(\vec{r}_3(t)) = 4t$
$t = 0$

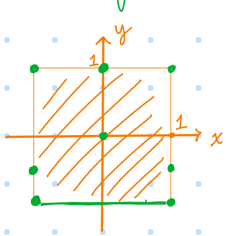
$\boxed{\begin{array}{l}\text{crit pt } (0,1) \\ \text{endpoints redundant}\end{array}}$

$f(\vec{r}_4(t)) = t^2 + 1 - t^2 + 4$
$= 5$
$\frac{d}{dt} f(\vec{r}_4(t)) = 0$

$\boxed{\begin{array}{l}\text{infinitely many crit pts} \\ (t,-1) \text{ for all } -1 \le t \le 1\end{array}}$

Step 3: Compare values of f at critical points

Picture of all critical points



$f(0,0) = 4$
$f(1,-\frac{1}{2}) = 4.75$
$f(1,-1) = 5$
$f(1,1) = 7$

$f(-1,-\frac{1}{2}) = 4.75$
$f(0,1) = 5$
$f(-1,1) = 7$
$f(-1,-1) = 5$

$f(t,-1) = 5$

$\boxed{\begin{array}{l}\text{Absolute max value of } 7 \text{ at } (1,1) \text{ and } (-1,1) \\ \text{Absolute min value of } 4 \text{ at } (0,0)\end{array}}$

ex 5) (variant of a question on WebAssign 11.7)

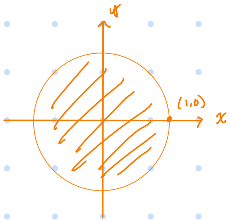
Find absolute extrema of $f(x,y) = 2x^3 + y^4$ on $D = \{(x,y) \mid x^2 + y^2 \le 1\}$

$f_x = 6x^2$
$f_y = 4y^3$

$\begin{cases} 6x^2 = 0 \\ 4y^3 = 0 \end{cases}$

• If $x=0$, $y=0$
$\boxed{\text{crit pt } (0,0)}$

Picture of D



Boundary of D

$\vec{r}(t) = \langle \cos t, \sin t \rangle$
$0 \le t \le 2\pi$

Critical points of boundary

$f(\vec{r}(t)) = 2\cos^3 t + \sin^4 t$

$\frac{d}{dt} f(\vec{r}(t)) = 6\cos^2 t(-\sin t) + 4\sin^3 t(\cos t)$

$0 = 2\sin t \cos t\,(-3\cos t + 2\sin^2 t)$

$\sin t = 0$
$t = 0, \pi$
$\boxed{\text{c.p. } (1,0) \ (-1,0)}$

$\cos t = 0$
$t = \frac{\pi}{2}, \frac{3\pi}{2}$
$\boxed{\text{c.p. } (0,1) \ (0,-1)}$

$-3\cos t + 2\sin^2 t = 0$
$-3\cos t + 2(1 - \cos^2 t) = 0$
$2\cos^2 t + 3\cos t - 2 = 0$
$(2\cos t - 1)(\cos t + 2) = 0$

$\cos t = \frac{1}{2}$
$t = \frac{\pi}{3}, \frac{5\pi}{3}$
$\boxed{\text{c.p. } \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right) \ \left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right)}$

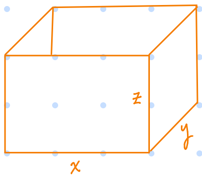
$\cos t = -2$
no soln

Compare values of f at critical points

(x,y)	$(0,0)$	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$f(x,y)$	0	2	1	-2	1	.8125	.8125
		↑ max		↑ min			

f has a max value of 2 at $(1,0)$
f has a min value of -2 at $(0,1)$

ex 5) [section 11.7 prob 47] A cardboard box without a lid is to have a volume of $32{,}000\ \text{cm}^3$. Find the dimensions that minimize the amount of cardboard used.



$$\text{Volume} = xyz$$
$$32{,}000 = xyz$$
$$z = \frac{32{,}000}{xy}$$

$$\text{Cardboard used} = xy + 2xz + 2yz$$
$$= xy + 2z(x+y)$$
$$= xy + 2\left(\frac{32{,}000}{xy}\right)(x+y)$$

$$0 < x < 16{,}000 \qquad 0 < y < 16{,}000$$

Goal: Find min of $f(x,y) = xy + 2(x+y)\left(\frac{32{,}000}{xy}\right)$ on $0 < x < 16{,}000,\ 0 < y < 16{,}000$.

$$f_x = y + 2\left(\frac{32{,}000}{xy}\right) + 2(x+y)\left(-\frac{32{,}000}{xy^2}\right)$$
$$f_y = x + 2\left(\frac{32{,}000}{xy}\right) + 2(x+y)\left(-\frac{32{,}000}{x^2y}\right)$$

$$\begin{cases} xy^3 + 64{,}000y - 64{,}000(x+y) = 0 \\ x^3y + 64{,}000x - 64{,}000(x+y) = 0 \end{cases}$$

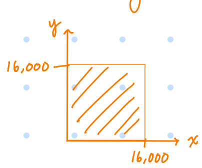
$$\begin{cases} xy^3 - 64{,}000x = 0 \\ x^3y - 64{,}000y = 0 \end{cases}$$

$$\begin{cases} x(y^3 - 64{,}000) = 0 \\ y(x^3 - 64{,}000) = 0 \end{cases}$$

- If $x=0$, $y=0$
- If $y^3 - 64000 = 0$, $y = 40$, $x = 40$

$\boxed{\begin{array}{l} \text{crit pt } (0,0) \\ \text{crit pt } (40,40) \end{array}}$

Picture of D



Boundary of D

$\vec{r}_1(t) = \langle 0, t \rangle \qquad 0 < t < 16{,}000$

$\vec{r}_2(t) = \langle t, 0 \rangle \qquad 0 < t < 16{,}000$

$\vec{r}_3(t) = \langle 16{,}000, t \rangle \qquad 0 < t < 16{,}000$

$\vec{r}_4(t) = \langle t, 16{,}000 \rangle \qquad 0 < t < 16{,}000$

Find critical points on boundary

$f(\vec{r}_1(t))$ undefined

$f(\vec{r}_2(t))$ undefined

$$f(\vec{r}_3(t)) = 16{,}000t - 64{,}000\left(\frac{16{,}000 + t}{16{,}000\,t}\right)$$
$$= 16{,}000t - 4\left(\frac{16{,}000}{t} + 1\right)$$
$$\frac{d}{dt} f(\vec{r}_3(t)) = 16{,}000 - 4\left(\frac{-16{,}000}{t^2}\right)$$
$$0 = 16{,}000 + \frac{64{,}000}{t^2}$$

no solution

$f(\vec{r}_4(t)) =$ same as $f(\vec{r}_3(t))$

$\boxed{\begin{array}{l} \text{endpoints: } (0,0),\ (0, 16\,000), \\ (16\,000, 0),\ (16\,000, 16\,000) \end{array}}$

Compare function values (use limits when f is undefined)

(x,y)	$(0,0)$	$(40,40)$	$(0, 16\,000)$	$(16\,000, 0)$	$(16\,000, 16{,}000)$
$f(x,y)$	limit goes to ∞	4800	limit goes to ∞	limit goes to ∞	2.56×10^8

$\boxed{\text{Let } x = 40\text{ cm},\ y = 40\text{ cm},\ z = 20\text{ cm to minimize cardboard.}}$

Wednesday October 7

Reminders

- Compile HW 7 for André

11.8 Lagrange Multipliers

Let's consider example 5 from yesterday from a new perspective

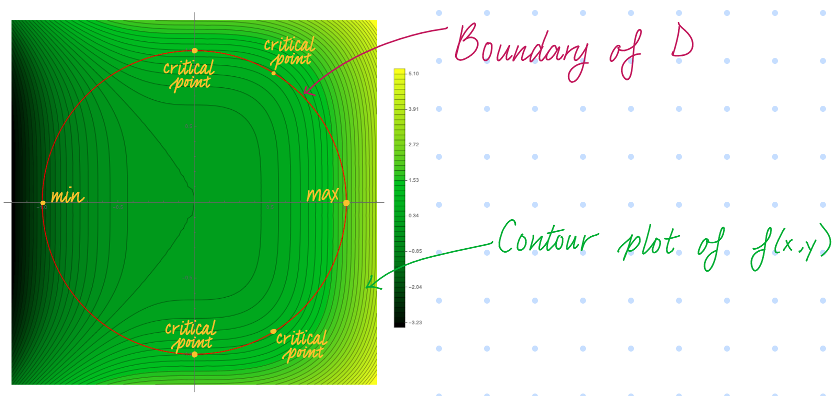
ex 5) Find absolute extrema of $f(x,y) = 2x^3 + y^4$ on $D = \{(x,y) \mid x^2 + y^2 \le 1\}$

There were 3 parts to this problem

1) Find critical points inside D by solving $\nabla f = \vec{0}$
2) Find critical points on boundary by parametrizing boundary, plugging into f, and solving $\frac{d}{dt} f(\vec{r}(t)) = 0$
3) Compare all function values at all critical points

We investigate step 2 graphically.

Can we visually find the max and min of f along the boundary?



Notice that the critical points occur where the boundary curve is tangent to a level curve in the contour plot.

> Method of Lagrange multipliers
>
> Goal: Find max/min values of $f(x,y,z)$ given some constraint $g(x,y,z) = k$
>
> Step 1: Solve for x, y, z, and λ using $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = k$
>
> Step 2: Compare the values of f at points found in step 1.

ex 1) Find the extrema of $f(x,y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$

$f(x,y) = x^2 + y$
$g(x,y) = x^2 + y^2$ (more than one valid choice for g)
constraint : $g(x,y) = 1$

$\nabla f = \langle 2x, 1 \rangle$
$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$
$\langle 2x, 1 \rangle = \lambda \langle 2x, 2y \rangle$
$\langle 2x, 1 \rangle = \langle 2x\lambda, 2y\lambda \rangle$

Solve the system $\begin{cases} 2x = 2x\lambda \\ 1 = 2y\lambda \\ x^2+y^2 = 1 \end{cases}$ ← Warning! Do not cancel the $2x$!

$\begin{cases} 2x - 2x\lambda = 0 \\ 2y\lambda = 1 \\ x^2+y^2 = 1 \end{cases}$

$\begin{cases} 2x(1-\lambda) = 0 \\ 2y\lambda = 1 \\ x^2+y^2 = 1 \end{cases}$

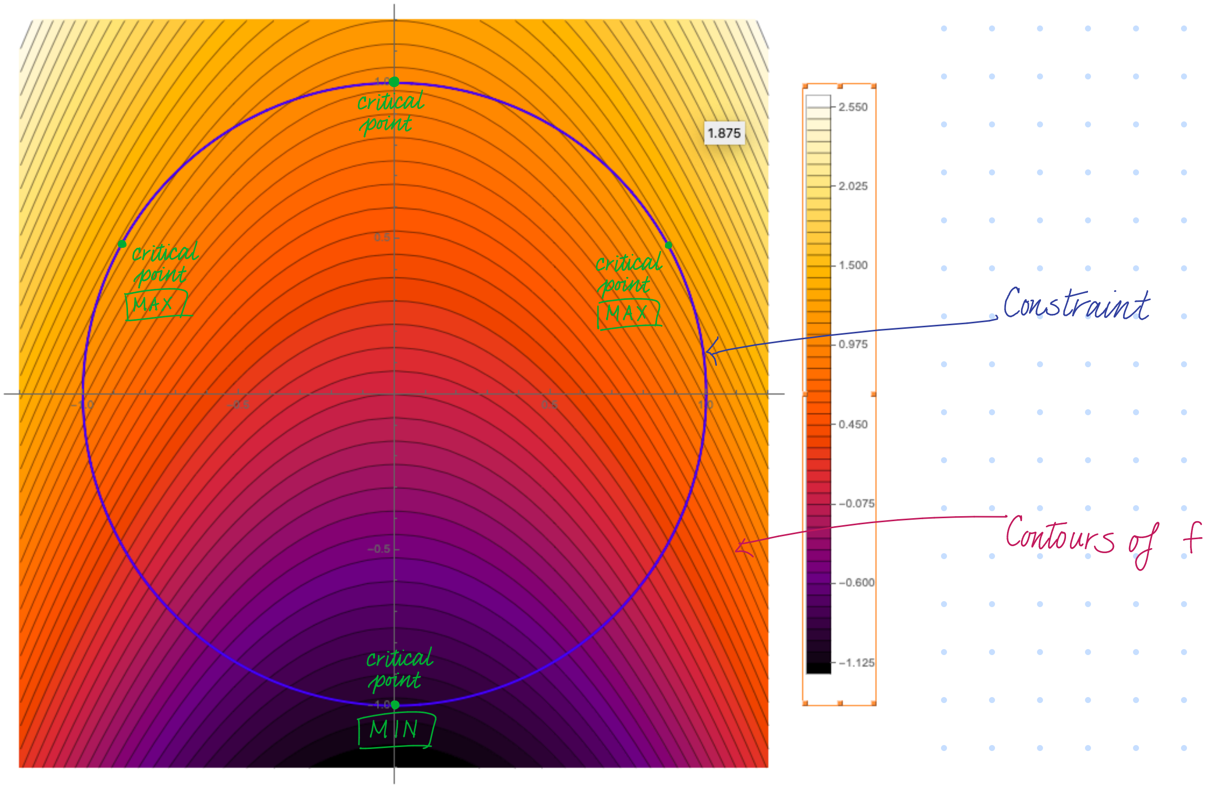
- If $x=0$, then use third equation to get $y = \pm 1$. Then use second equation to get λ. crit pts $(0,1)$ $\lambda = \frac{1}{2}$; $(0,-1)$ $\lambda = -\frac{1}{2}$
- If $1-\lambda = 0$, then $\lambda = 1$. Use second equation to get $y = \frac{1}{2}$. Then use third equation to get $x = \frac{\pm\sqrt{3}}{2}$. crit pts $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ $\lambda = 1$; $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$ $\lambda = 1$
- If $2y\lambda = 1$, $\lambda = \frac{1}{2y}$. Plug in to first equation to get $x(2y-1) = 0$. This only gives redundant points

Test critical points

(x,y)	$(0,1)$	$(0,-1)$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$f(x,y)$	1	-1	$5/4$	$5/4$

When restricted to $g(x,y) = 1$, f has a max of $5/4$ at $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and a min of -1 at $(0,-1)$

ex 2) Find the extrema of $f(x,y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$ by examining the contour plot



ex3) [section 11.8 prob 30] Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

Goal: Find min of $d(x,y,z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$ given x, y, z must satisfy the constraint $y^2 = 9 + xz$

(Pro tip: minimize $D = d^2 = x^2 + y^2 + z^2$ instead for easier algebra)

$D(x,y,z) = x^2 + y^2 + z^2$
$g(x,y,z) = y^2 - xz - 9$
constraint : $g(x,y,z) = 0$

$$\nabla D = \lambda \nabla g$$
$$\langle 2x, 2y, 2z \rangle = \lambda \langle -z, 2y, -x \rangle$$

Solve $\begin{cases} 2x = -\lambda z \\ 2y = 2\lambda y \\ 2z = -\lambda x \\ y^2 - xz - 9 = 0 \end{cases}$ $\quad \begin{cases} 2x = -\lambda z \\ 2y(1-\lambda) = 0 \\ 2z = -\lambda x \\ y^2 - xz - 9 = 0 \end{cases}$

- Solve for λ in first and third equations to get $\frac{2x}{z} = \frac{2z}{x}$, so $x^2 = z^2$
- Equation 2 shows that $y = 0$ is possible. This gives $xz = -9$ in equation 4. Together with $x^2 = z^2$, we get these critical points

$(3,0,3)$	$\lambda = -2$
$(3,0,-3)$	$\lambda = 2$
$(-3,0,3)$	$\lambda = 2$
$(-3,0,-3)$	$\lambda = -2$

- Equation 2 shows $\lambda = 1$ is possible. Plug this in to first and third equation to get $x = 0, z = 0$.

$(0,3,0)$	$\lambda = 1$
$(0,-3,0)$	$\lambda = 1$

Compare values at critical points

$D(3,0,3) = 18$
$D(3,0,-3) = 18$
$D(-3,0,3) = 18$
$D(-3,0,-3) = 18$
$D(0,3,0) = 9$
$D(0,-3,0) = 9$

> The points $(0, \pm 3, 0)$ on the surface $y^2 = xz + 9$ are the closest points to the origin. They are $\sqrt{9} = 3$ units from the origin.

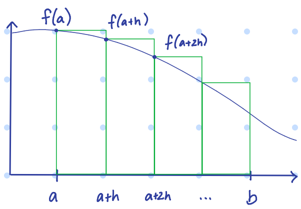
Friday October, 9

Reminders

- All Week 7 WebAssign due before midnight on Sunday
- HW 8 sections 11.8, 12.1, $\boxed{A1}$, A2 super helpful for Quiz 4 😉
- Study for Quiz 4 (sections 11.5 to 12.1)
- Review Calc 2 concept: Area between curves (links in Piazza)

12.1 Double integrals over rectangles

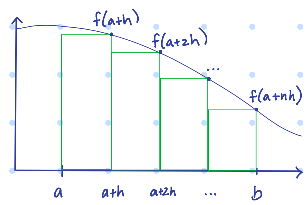
Area under a curve



Left endpoints with n rectangles

Area $\approx f(a)\cdot h + f(a+h)\,h + \cdots f(a+(n-1)h)\cdot h$

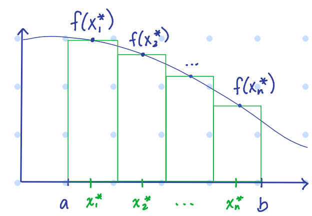
$$L_n = \sum_{i=0}^{n-1} h\, f(a+ih)$$



Right endpoints with n rectangles

Area $\approx f(a+h)\,h + f(a+2h)\,h + \cdots f(a+nh)\,h$

$$R_n = \sum_{i=1}^{n} h\, f(a+ih)$$



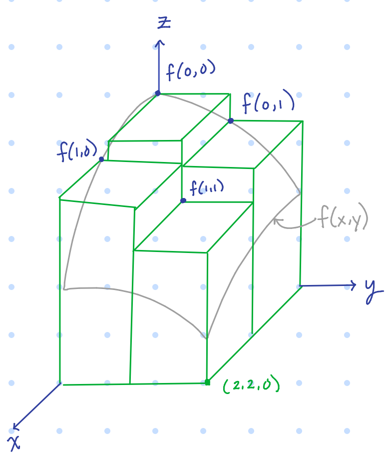
Midpoints with n rectangles

Area $\approx f(x_1^*)\,h + f(x_2^*)\,h + \cdots + f(x_n^*)\,h$

$$M_n = \sum_{i=1}^{n} h\, f(x_i^*)$$

Key idea: Take the limit as the width of rectangles goes to zero to get $\int_a^b f(x)\,dx$

Volume under a surface



In this drawing, the height of each box is given by the function's value at the corner closest to the origin. Like in the case for area, using a different point (x,y) in the domain will give a different volume estimate

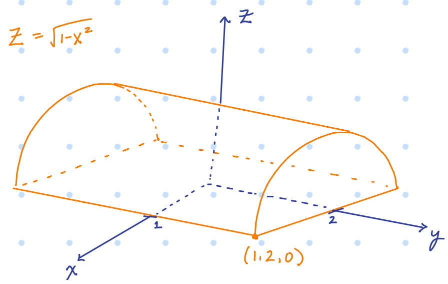
$$\text{Volume} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

, where ΔA is the area of the base of each box, ie the part on the xy-plane

Key idea: True volume under $z = f(x,y)$ $= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$

> Definition: The double integral of f over the region R is
>
> $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

ex 1) Evaluate $\iint_R \sqrt{1-x^2}\, dA$ for $R = [-1,1] \times [-2,2]$ by interpreting it geometrically



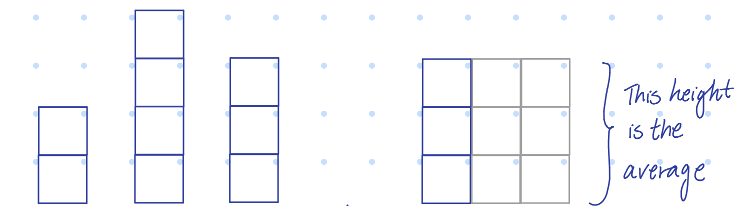
$\iint_R \sqrt{1-x^2}\, dA$ is the volume of half a cylinder of radius 1, height 4

$$\iint_R \sqrt{1-x^2}\, dA = \frac{1}{2}\pi r^2 h$$
$$= \frac{1}{2}\pi (1)^2 (4)$$
$$= \boxed{2\pi}$$

I wrote 2 during class but it should be 4

Average value

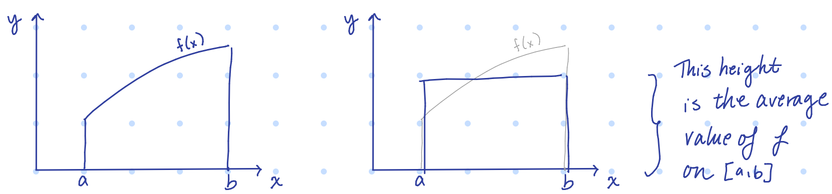
average value of numbers



$(2 + 4 + 3)/3 = 3$

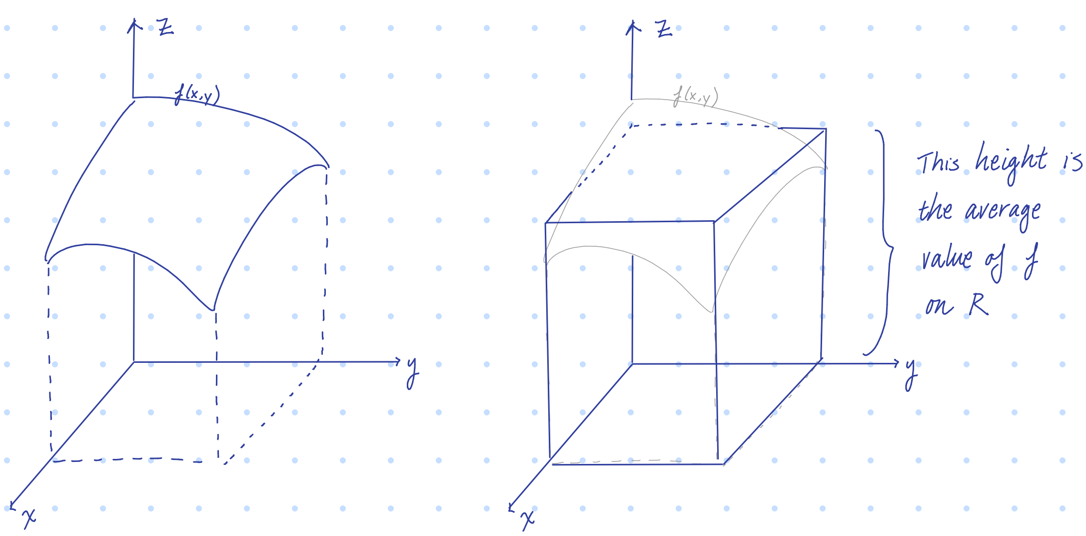
$$\text{average} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

average value of $y = f(x)$ on $[a,b]$



$$\text{average} = \frac{\int_a^b f(x)\, dx}{b-a}$$

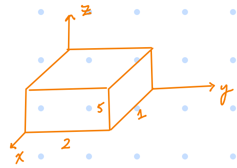
average value of $z = f(x,y)$ on region R



$$\text{average} = \frac{\iint_R f(x,y)\, dA}{\text{Area of } R}$$

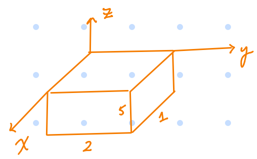
ex2) Evaluate each integral over the region $R = [0,1] \times [0,2]$ by interpreting it geometrically

(a) $\iint_R 5 \, dA$



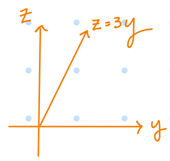
$\boxed{10}$

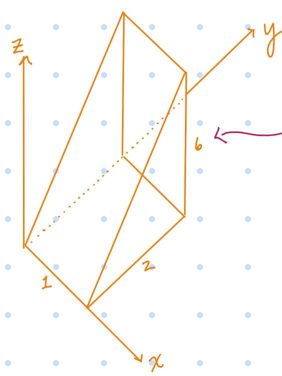
(b) $\iint_R -5 \, dA$



$\boxed{-10}$

(c) $\iint_R 3y \, dA$

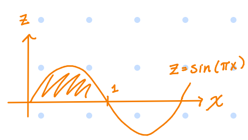


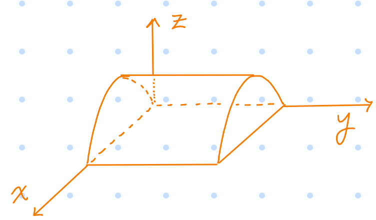


I wrote 3 in class but it should be 6.

$\boxed{6}$

(d) $\iint_R \sin(\pi x) \, dA$

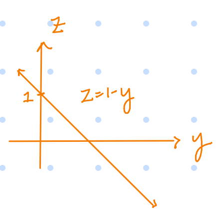


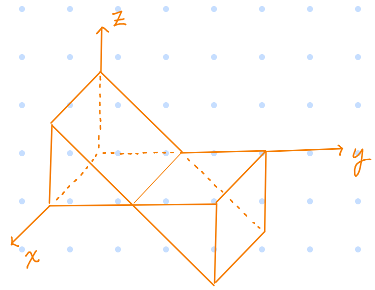


Area of one "hump" of sin curve $= \int_0^1 \sin \pi x \, dx = \frac{2}{\pi}$

$\boxed{4/\pi}$

(e) $\iint_R 1-y \, dA$





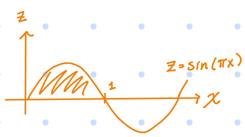
$\boxed{0}$

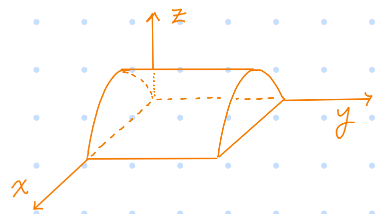
12.2 Iterated integrals

How do you compute $\iint_R f(x,y)\,dA$ without interpreting it geometrically?

Example 2(d) in 12.1 gives us a hint

2(d) $\iint_R \sin(\pi x)\,dA$





Area of one "hump" of sin curve $= \int_0^1 \sin \pi x\,dx = \frac{2}{\pi}$

$\boxed{4/\pi}$

In this example, we computed the area of one slice by doing $\int_0^1 \sin(\pi x)\,dx$, and then "dragging" the slice along the y-axis by 2 units to fill the desired volume. In other words,

$$\iint_R \sin(\pi x)\,dx = \int_0^2 \left(\int_0^1 \sin(\pi x)\,dx \right) dy$$

ex 1) $\int_0^3 \int_1^2 x^2 y\,dy\,dx = \int_0^3 x^2 \left(\int_1^2 y\,dy \right) dx$

$= \int_0^3 x^2 \left[\frac{1}{2} y^2 \right]_{y=1}^{y=2} dx$

$= \int_0^3 \frac{3}{2} x^2\,dx$

$= \left[\frac{1}{2} x^3 \right]_{x=0}^{x=3}$

$= \boxed{27/2}$

ex 2) $\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_1^2 \left[\frac{1}{3} x^3 y \right]_{x=0}^{x=3} dy$

$= \int_1^2 9y\,dy$

$= \left[\frac{9}{2} y^2 \right]_{y=1}^{y=2}$

$= \boxed{27/2}$

Fubini's Theorem

If $f(x,y)$ is continuous on the rectangle $[a,b] \times [c,d]$, then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Monday October 12

Reminders

- Study for Quiz 4 (practice problems posted in Resources)
- WebAssign 12.2
- HW 8, section 12.2 and problem A2

12.2 Iterated integrals (cont)

Do warm-up Zoom poll

ex3) Evaluate $\iint_R y\sin(xy)\,dA$ where $R=[1,2]\times[0,\pi]$.

Is $\iint y\sin(xy)\,dy\,dx$ or $\iint y\sin(xy)\,dy\,dx$ easier to compute?

looks like integration by parts, eek!

looks simple

Integrating dydx

$$\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx$$

$u=y \qquad dv=\sin(xy)\,dy$

$du=dy \qquad v=-\frac{\cos(xy)}{x}$

$$\int_1^2\left(\left[\frac{-y\cos(xy)}{x}\right]_{y=0}^{y=\pi}-\int_0^\pi -\frac{1}{x}\cos(xy)\,dy\right)dx$$

$$\int_1^2\left(-\frac{\pi}{x}\cos(\pi x)+\frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}\right)dx$$

$$\int_1^2\left(-\frac{\pi}{x}\cos(\pi x)+\frac{1}{x^2}\sin(\pi x)\right)dx$$

$u=-\frac{1}{x} \qquad dv=\pi\cos\pi x\,dx$

$du=\frac{1}{x^2}\,dx \qquad v=\sin\pi x$

$$\left[\frac{-1}{x}\sin\pi x\right]_1^2-\int_1^2\frac{1}{x^2}\sin\pi x\,dx+\int_1^2\frac{1}{x^2}\sin\pi x\,dx$$

$$-\frac{1}{2}\sin(2\pi)+\sin\pi=\boxed{0}$$

Integrating dxdy

$$\int_0^\pi\int_1^2 y\sin(xy)\,dx\,dy$$

$$\int_0^\pi\left[-\cos(xy)\right]_{x=1}^{x=2}dy$$

$$\int_0^\pi -\cos(2y)+\cos(y)\,dy$$

$$\left[-\frac{1}{2}\sin(2y)+\sin(y)\right]_0^\pi$$

$$\boxed{0}$$

Key idea: pick easy order of integration whenever possible.

ex 4) [section 12.2 prob 23, closely related to HW prob # 24]

Sketch the solid whose volume is given by the iterated integral $\int_0^1\int_0^1 4-x-2y\,dx\,dy$

The function $z=4-x-2y$ is a plane.

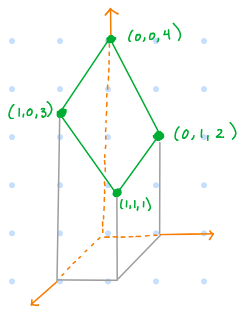
The domain of integration is a square, so plug in the corners of the square to help sketch.

$(0,0)$ $z=4$

$(0,1)$ $z=2$

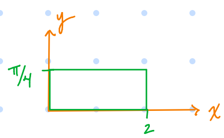
$(1,0)$ $z=3$

$(1,1)$ $z=1$



ex 4) Find the volume of the solid enclosed by the surface $z=x\sec^2 y$ and the planes $z=0$, $x=0$, $x=2$, $y=0$, and $y=\pi/4$

Domain of integration



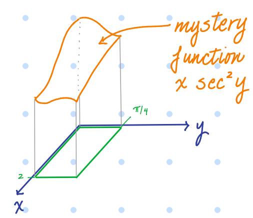
I know $x\sec^2 y \geq 0$ on this domain, even though I don't know what the graph looks like.

$$\int_0^2\int_0^{\pi/4} x\sec^2 y\,dy\,dx = \int_0^2 x\left[\tan y\right]_{y=0}^{y=\pi/4}dx$$

$$= \int_0^2 x\,dx$$

$$= \boxed{2}$$

Concept:

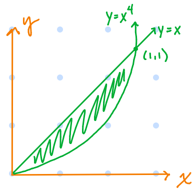


12.3 Double integrals over general regions

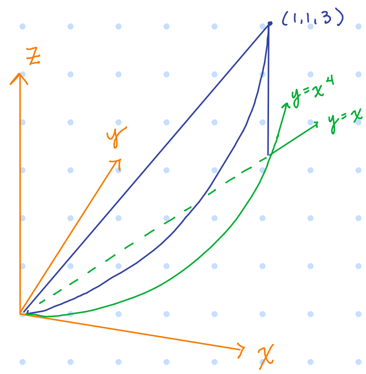
If the domain of integration is more complicated than a rectangle, then we may need functions to serve as the bounds of the integral instead of using numbers only.

ex 1) Find the volume of the solid under the plane $x+2y-z=0$ and above $z=0$, bounded by $y=x$ and $y=x^4$.

Domain of integration :



The solid



Function values at corners : $(0,0)$ $z=0$

$(1,1)$ $z=3$

$$\text{Volume} = \int_0^1\int_{x^4}^{x} x+2y \; dy\,dx$$

$$= \int_0^1 \left[xy+y^2\right]_{y=x^4}^{y=x} dx$$

$$= \int_0^1 (x^2+x^2)-(x^5+x^8)\,dx$$

(check: After completing the integral with respect to y, there should only be x's left.)

$$= \int_0^1 2x^2-x^5-x^8\,dx$$

$$= \left[\tfrac{2}{3}x^3-\tfrac{1}{6}x^6-\tfrac{1}{9}x^9\right]_0^1$$

$$= \tfrac{12}{18}-\tfrac{3}{18}-\tfrac{2}{18}$$

$$= \boxed{7/18}$$

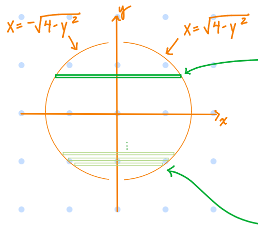
(check: Since we are computing a definite integral, your final answer must be a number)

Note : The other order of integration will also work

$$\int_0^1\int_y^{y^{1/4}} x+2y\;dx\,dy = \boxed{7/18}$$

ex 2) Evaluate $\iint_D 1\, dA$ for the region $D = \{(x,y) : x^2+y^2 \le 4\}$

Domain of integration:



A typical "slice" of the domain has an x-coordinate that starts at $-\sqrt{4-y^2}$ and ends at $\sqrt{4-y^2}$.

The slices themselves have y-coordinates from -2 to 2

The integral

$$\iint_D 1\, dA = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 1 \; dx\, dy \qquad (1)$$

$$= \int_{-2}^{2} \Big[\, x \,\Big]_{x=-\sqrt{4-y^2}}^{x=\sqrt{4-y^2}} dy \qquad (2)$$

$$= \int_{-2}^{2} \sqrt{4-y^2} - \left(-\sqrt{4-y^2}\right) dy \qquad (3)$$

$$= \int_{-2}^{2} 2\sqrt{4-y^2}\; dy \qquad (4)$$

$$= \boxed{4\pi}$$

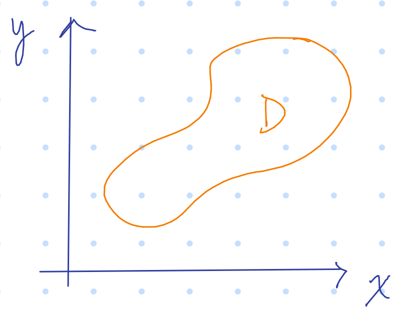
Ponder this: Why is the answer to example 2 equal to the area of the circle?

Notice that in equation (3), after plugging in our bounds for x, we have $\int_{\text{bottom}}^{\text{top}} (\text{right fn}) - (\text{left fn})\, dy$

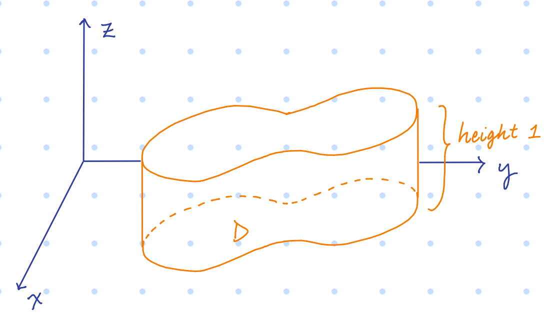
This is exactly how we computed areas between curves in single-variable calc!

The integral $\iint_D 1\, dA$ has two geometric meanings

- $\iint_D 1\, dA$ = area of the region D

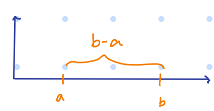


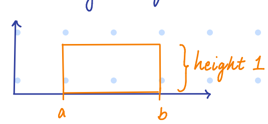
- $\iint_D 1\, dA$ = volume of "cylinder" with base D and height 1.



This principle also held in single-variable calc

$\int_a^b 1\, dx = b - a$ = length of interval $[a,b]$ = area under $y=1$ from $x=a$ to $x=b$



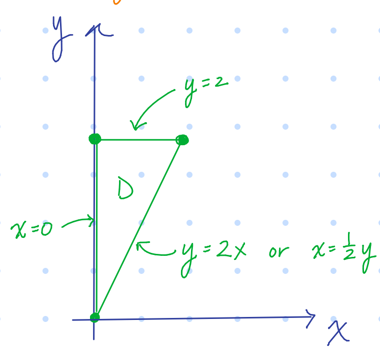


Thought for sleepless nights: What if the domain of integration is a 3-dimensional solid? 🤔

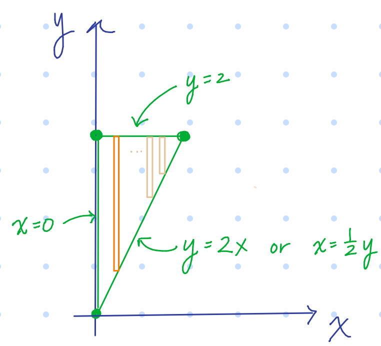
(coming soon in 12.7)

ex 3) Evaluate the double integral $\iint_D \sin(y^2)\, dA$ for the triangular region D with vertices $(0,0)$, $(0,2)$, and $(1,2)$.
Write two integrals, one for each order of integration, and evaluate by picking the more convenient order.

Domain of integration



Integrating dy dx

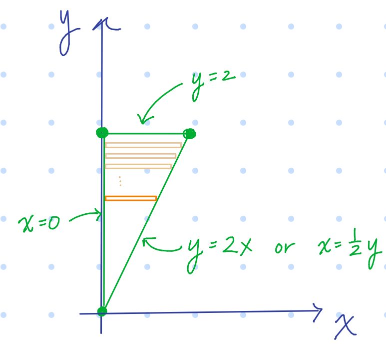


A typical slice has a y-coordinate from $2x$ to 2

The x-coordinate of the first slice is 0 and the last slice is at $x=1$

$$\int_0^1 \int_{2x}^2 \sin(y^2)\; dy\, dx$$

Integrating dx dy



A typical slice has an x-coordinate from 0 to $\frac{1}{2}y$

The y-coordinate of the first slice is 0 and the last slice is at $y=2$

$$\int_0^2 \int_0^{\frac{1}{2}y} \sin(y^2)\; dx\, dy$$

[This one is easier to compute
Do you see why?]

Evaluating the integral

$$\int_0^2 \int_0^{\frac{1}{2}y} \sin(y^2)\; dx\, dy = \int_0^2 \left[x \sin(y^2)\right]_{x=0}^{x=\frac{1}{2}y} dy$$

$$= \int_0^2 \frac{1}{2} y \sin(y^2)\; dy$$

$$u = y^2, \quad du = 2y\, dy$$

$$= \int_0^4 \frac{1}{4} \sin(u)\; du$$

$$= -\frac{1}{4}\left[\cos u\right]_{u=0}^{u=4}$$

$$= \boxed{-\frac{1}{4}\left(\cos 4 - 1\right)}$$

Tuesday October 13

Reminders

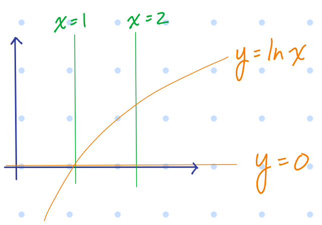
- [Important!] Do Quiz 4 tonight, 7-10 pm
  - Set a timer
  - Proctorio check

12.3 Double integrals over general regions (cont)

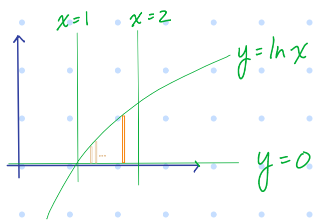
ex 4) Sketch the region of integration and change the order of integration

$$\int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx$$

Draw $x=1$, $x=2$, $y=0$, $y=\ln x$



Current order of integration

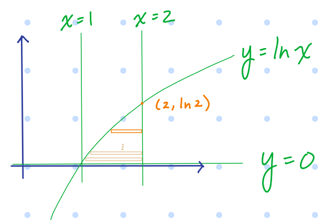


$$\int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx$$

A typical slice has y-coord starting at 0 going to $\ln x$

The slices start with x-coord 1 and end with x-coord 2

Desired order of integration



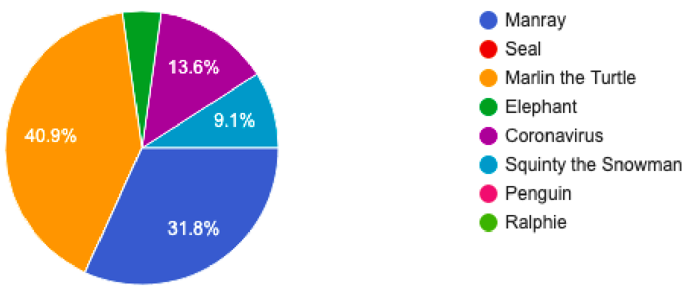
$$\boxed{\int_0^{\ln 2} \int_{e^y}^{2} f(x,y)\, dx\, dy}$$

Wednesday October 14

Groups will be changing

Reminders

- Compile HW8 for André
- Submit Quiz 4 by 10 pm (not midnight!)
- WebAssign 12.3, HW9 section 12.3



Marlin seems very chill. I think he'd be a good hang

They are all great!

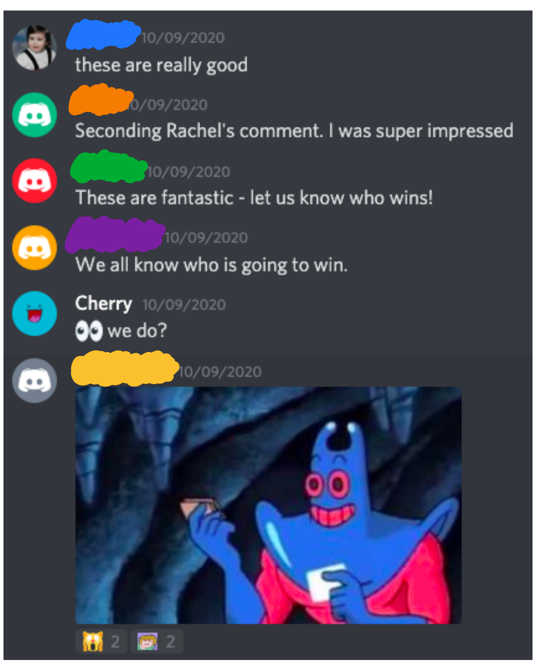
The penguin was also awesome, and oh! the Corona virus made me laugh, but Marlin the Turtle wins for intricacy + making me laugh

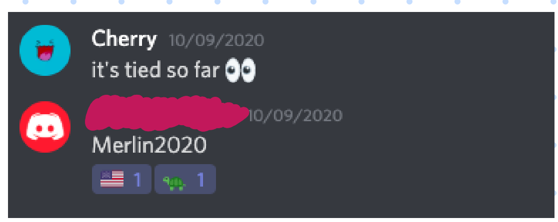
These graphs were so cool!!!!!

They were all so good it's impossible to actually choose my favorite!

😝

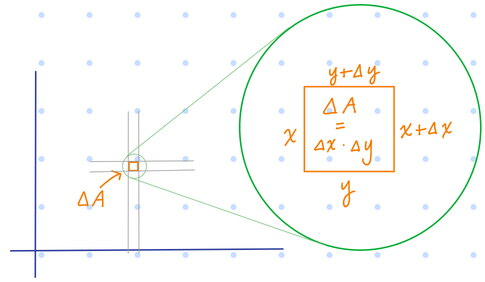
muh-muh muh-muh, muh-muh-muh, muh-muh



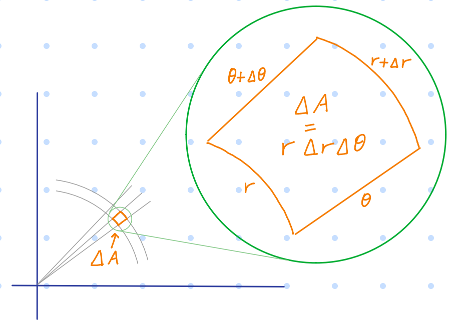


12.4 Double integrals in polar coordinates

For a double integral $\iint_D f \, dA$, the "dA" represents an infinitesimal bit of area. In Cartesian coordinates, a tiny increase in x by Δx and a tiny increase in y by Δy gives a tiny patch of area ΔA equal to $\Delta x \cdot \Delta y$. So it is sensible that $\boxed{dA = dy\,dx \text{ or } dx\,dy}$



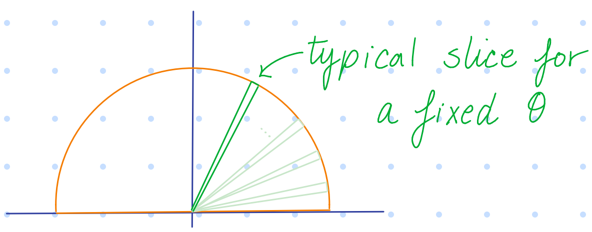
But in polar coordinates, a tiny increase in r and in θ doesn't form a rectangle! So if you want to use polar coordinates, $dA \neq dr\,d\theta$. The correct interpretation of dA in polar is $\boxed{dA = r\,dr\,d\theta}$



$\text{Area} = \left[\pi(r+\Delta r)^2 - \pi r^2\right]\frac{\Delta\theta}{2\pi}$
$= \ldots$
$= r\,\Delta r\,\Delta\theta$

How do we write bounds for an integral in polar coordinates?

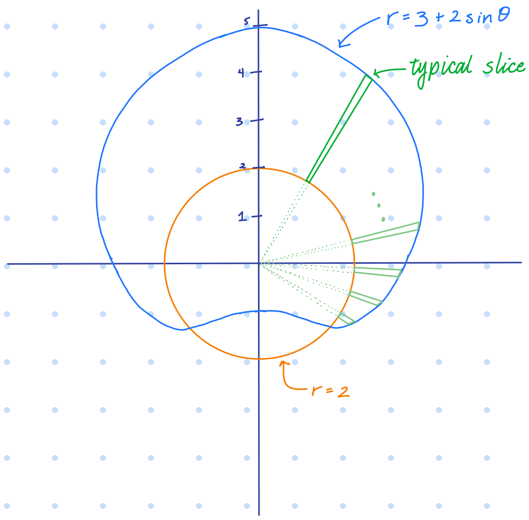
ex 1) Convert $\iint_D f(x,y)\,dA$ to polar coordinates for the domain D.
D is the disk of radius 3 in the upper half plane.



$$\boxed{\int_0^{\pi}\int_0^3 f(r\cos\theta,\, r\sin\theta)\; r\,dr\,d\theta}$$

ex 2) Convert $\iint_D f(x,y)\,dA$ to polar coordinates for the domain D.

D is the region inside $r = 3+2\sin\theta$ and outside $r=2$, depicted below.



Find intersection of curves

$2 = 3 + 2\sin\theta$

$-\frac{1}{2} = \sin\theta$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Write integral

$$\boxed{\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \int_{2}^{3+2\sin\theta} f(r\cos\theta, r\sin\theta)\; r\,dr\,d\theta}$$

ex 3) Convert $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\; dy\,dx$ to polar and evaluate

Domain of integration

$x=0,$

$x=2$

$y=0$

$y=\sqrt{2x-x^2} \Rightarrow y^2 = 2x-x^2$

$y^2+x^2-2x=0$

$y^2+x^2-2x+1=1$

$y^2+(x-1)^2=1$

circle at $(1,0)$, $r=1$

convert $y^2+(x-1)^2=1$ to polar

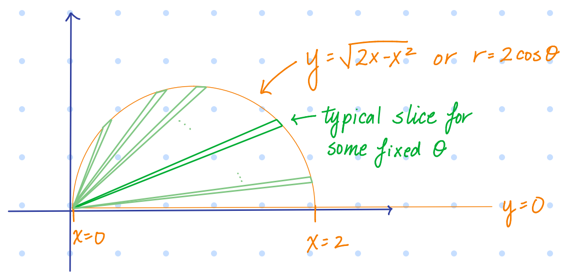
$r^2\sin^2\theta + r^2\cos^2\theta - 2r\cos\theta = 0$

$r^2 - 2r\cos\theta = 0$

$r(r-2\cos\theta)=0$

$r=0$ is a single point and clearly not the circle we want

$\boxed{r = 2\cos\theta}$



$$\underbrace{\int_0^{\pi/2}}\ \underbrace{\int_0^{2\cos\theta}} r \cdot \underbrace{r\,dr\,d\theta}$$

interpretation of "dA" in polar

A typical slice for fixed θ has r-coordinate starting at 0 going out to $2\cos\theta$

The first slice is given by $\theta=0$ and the last by $\theta = \pi/2$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^{2\cos\theta} d\theta$$

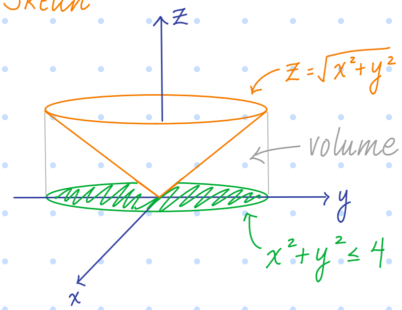
$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\; d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos\theta(1-\sin^2\theta)\,d\theta$$

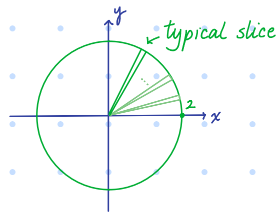
$$= \frac{8}{3}\int_0^1 1-u^2\,du = \boxed{\frac{16}{9}}$$

ex 4) Find the volume under the cone $z=\sqrt{x^2+y^2}$ and above the disk $x^2+y^2 \le 4$

Sketch



Domain of integration



In rectangular : $\iint_D \sqrt{x^2+y^2} - 0 \, dA$

In polar : $\int_0^{2\pi}\int_0^2 r \cdot r\,dr\,d\theta$

Evaluate $\int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta$ (1)

$= \int_0^{2\pi} \left[\frac{1}{3}r^3\right]_0^2 d\theta$ (2)

$= \int_0^{2\pi} \frac{8}{3}\,d\theta$ (3)

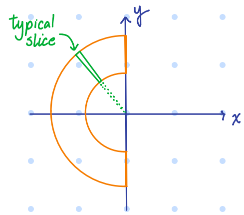
$= \boxed{\frac{16\pi}{3}}$

Computational trick: Notice how $\int_0^{2\pi} 1\,d\theta = 2\pi$? In equation (1), observe that the inner integral $\int_0^2 r^2 dr$ will not contain θ at all. We can use this fact as a shortcut.

$$\int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta = 2\pi \int_0^2 r^2\,dr$$

ex 5) Evaluate $\iint_R (x+y)\, dA$ where R is the region to the left of the y-axis between $x^2+y^2=1$ and $x^2+y^2=4$

Domain of integration



The integral

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$\int_{\pi/2}^{3\pi/2} \int_1^2 r^2(\cos\theta + \sin\theta)\, dr\, d\theta$$

$$\int_{\pi/2}^{3\pi/2} \left[\frac{1}{3} r^3 (\cos\theta + \sin\theta)\right]_1^2 d\theta$$

$$\frac{7}{3} \int_{\pi/2}^{3\pi/2} \cos\theta + \sin\theta\, d\theta$$

$$\frac{7}{3}\left[\sin\theta - \cos\theta\right]_{\pi/2}^{3\pi/2}$$

$$\frac{7}{3}\left[-1-(-1)\right] = \boxed{\frac{-14}{3}}$$

Friday October 16

Reminders

- All Week 8 WebAssign due Sun before midnight
- Written HW 9, section 12.4 and A1, A2
- Review equations of surfaces

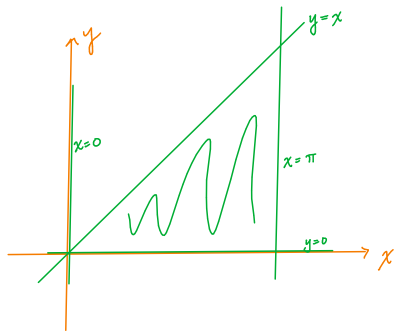
General advice for double integrals

- Draw the domain of integration
- Draw the 3D view, if applicable
- The inner bounds describe a typical slice

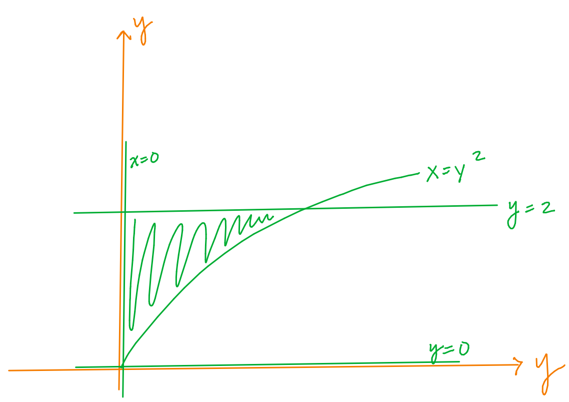
  The outer bounds describe where the slices stop and start

Name: ______________________________

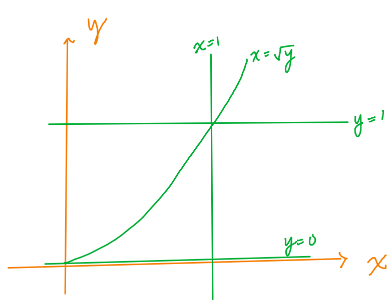
1. Sketch the region of integration for $\int_0^{\pi}\int_0^{x} y\sin x\; dy\; dx$.



2. Sketch the region of integration for $\int_0^{2}\int_0^{y^2} y^2 x\; dx\; dy$.



3. Evaluate the following integral by reversing the order of integration: $\int_0^{1}\int_{\sqrt{y}}^{1} \sqrt{2+x^3}\; dx\; dy$



$$\int_0^1\int_0^{x^2}\sqrt{2+x^3}\;dy\;dx$$

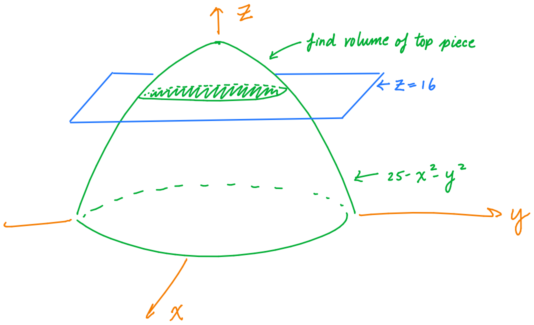
$$\int_0^1\left[y\sqrt{2+x^3}\right]_{y=0}^{y=x^2}dx$$

$$\int_0^1 x^2\sqrt{2+x^3}\;dx \qquad u=2+x^3,\; du=3x^2dx$$

$$\frac{1}{3}\int_2^3\sqrt{u}\;du$$

$$\boxed{\frac{2\sqrt{3}}{3}-\frac{4\sqrt{2}}{9}}$$

4. Set up, but do not evaluate, an iterated integral for the volume below the graph of $f(x,y) = 25 - x^2 - y^2$ and above the plane $z = 16$.



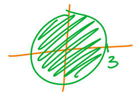
At $z=16$, the paraboloid's trace is a circle. We find its radius

$16 = 25 - x^2 - y^2$

$9 = x^2 + y^2$

radius $= 3$

Domain of integration



$$\underset{\substack{\text{circle of}\\ \text{radius } 3}}{\iint} 25 - x^2 - y^2 \, dA$$

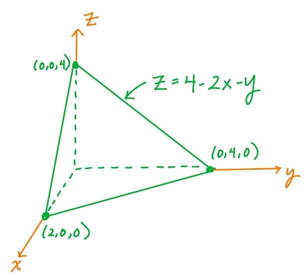
$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 25 - x^2 - y^2 \, dy \, dx$$

(polar is better!)

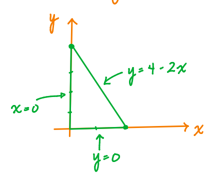
$$\int_0^{2\pi} \int_0^3 (25 - r^2) \, r \, dr \, d\theta = \boxed{\frac{369\pi}{2}}$$

5. Find the volume under the graph of $2x + y + z = 4$ in the first octant.

$z = 4 - 2x - y$



Domain of integration



$$\int_0^2 \int_0^{4-2x} 4 - 2x - y \, dy \, dx \quad \text{or} \quad \int_0^4 \int_0^{\frac{4-y}{2}} 4 - 2x - y \, dx \, dy$$

$\vdots$

$$\boxed{16/3}$$

6. Evaluate the integral $\displaystyle\iint_D xy \, dA$, where D is the disk with center $(0,0)$ and radius 3, by changing to polar coordinates.

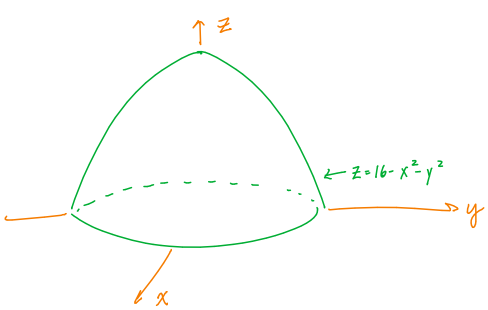
$$\int_0^{2\pi} \int_0^3 (r\cos\theta)(r\sin\theta) \, r \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^3 r^3 \cos\theta \sin\theta \, dr \, d\theta$$

$\vdots$

$$\boxed{0}$$

7. Use polar coordinates to find the volume of the solid below the paraboloid $x^2 + y^2 + z = 16$ and above the xy-plane.



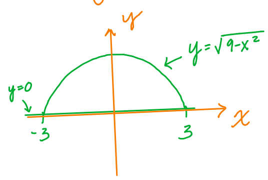
$$\int_0^{2\pi} \int_0^4 (16 - r^2) \, r \, dr \, d\theta$$

$\vdots$

$$\boxed{128\pi}$$

8. Evaluate the integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$ by converting to polar coordinates.

Domain of integration



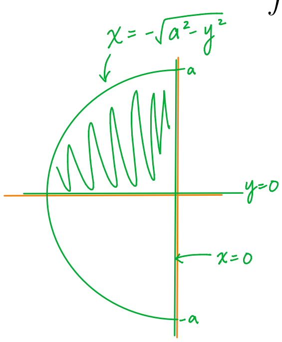
$$\int_0^{\pi}\int_0^{3} \sin(r^2)\cdot r\,dr\,d\theta \qquad \left(\begin{matrix} u=r^2 \\ du=2r\,dr\end{matrix}\right)$$

$$\frac{1}{2}\int_0^{\pi}\int_0^{9} \sin(u)\,du\,d\theta$$

$$\frac{1}{2}\int_0^{\pi}\Big[-\cos u\Big]_{u=0}^{u=9}\,d\theta$$

$$\boxed{\frac{\pi}{2}\left(1-\cos(9)\right)}$$

9. Evaluate the integral $\int_0^{a}\int_{-\sqrt{a^2-y^2}}^{0} x^2y\,dx\,dy$ by converting to polar coordinates.



$$\int_{\pi/2}^{\pi}\int_0^{a} r^2\cos^2\theta\cdot r\sin\theta\cdot r\,dr\,d\theta$$

$$\int_{\pi/2}^{\pi}\int_0^{a} r^4\cos^2\theta\sin\theta\,dr\,d\theta$$

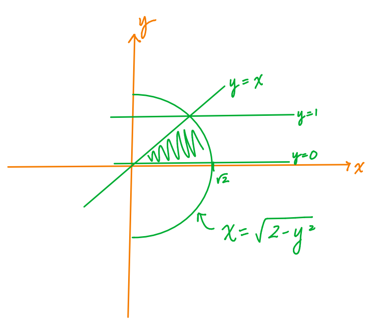
$$\int_{\pi/2}^{\pi}\cos^2\theta\sin\theta\left[\tfrac{1}{5}r^5\right]_{r=0}^{r=a}d\theta \qquad \tfrac{1}{3}u^3\Big]_{-1}^{0}$$

$$\frac{a^5}{5}\int_{\pi/2}^{\pi}\cos^2\theta\sin\theta\,d\theta \qquad \left(\begin{matrix} u=\cos\theta \\ du=-\sin\theta\,d\theta\end{matrix}\right)$$

$$-\frac{a^5}{5}\int_0^{-1}u^2\,du = \boxed{\frac{a^5}{15}}$$

10. Evaluate the integral $\int_0^{1}\int_{y}^{\sqrt{2-y^2}} (x+y)\,dx\,dy$ by converting to polar coordinates.

$x=\sqrt{2-y^2}$
$x^2=2-y^2$
$x^2+y^2=2$



$$\int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta+r\sin\theta)\,r\,dr\,d\theta$$

$$\int_0^{\pi/4}\int_0^{\sqrt{2}} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$\int_0^{\pi/4}(\cos\theta+\sin\theta)\left[\tfrac{1}{3}r^3\right]_{r=0}^{r=\sqrt{2}}d\theta$$

$$\frac{2\sqrt{2}}{3}\int_0^{\pi/4}\cos\theta+\sin\theta\,d\theta = \boxed{\frac{2\sqrt{2}}{3}}$$

11. (Challenge) Evaluate the integral $\int_0^{2}\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$ by converting to polar coordinates.

Check-in 13

Reverse the order of integration for $\int_0^1 \int_0^x f(x,y)\, dy\, dx$

Monday October 19

Reminders
- WebAssign 12.5
- HW 9, section 12.5
- Study for Check-in 14 (use Oct 16 worksheet)

12.5 Applications of double integrals

The double integral $\iint_D f\,dA$ has many interpretations. Today we will summarize some important ones.

1) $\iint_D f\,dA$ is the volume under the surface $z=f(x,y)$ over the region D (fig. 1)

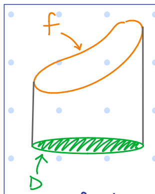


fig. 1

2) If $f(x,y)=1$, then $\iint_D 1\,dA =$ area of D

3) $\boxed{\text{If } f(x,y) = \text{density of something at a point } (x,y), \text{ then } \iint_D f\,dA = \text{total amount of that thing.}}$ ← new idea

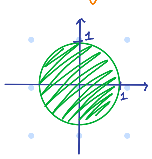
ex 1) If $\rho(x,y)$ is the population density at point (x,y), then the total population on a region D is $\iint_D \rho(x,y)\,dA$

ex 2) If $\rho(x,y)$ is the ~~population~~ charge density at point (x,y), then the total ~~population~~ charge on a region D is $\iint_D \rho(x,y)\,dA$

ex 3) Let $\rho(x,y)=y^2$ be the mass density, measured in g/cm^2, of D, a metal disk of radius 1 cm centered at the origin.

a) Compute the mass of the disk. Include units

Domain of integration



$$\text{mass} = \iint_D \rho(x,y)\,dA$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2\,dy\,dx \quad \underline{\text{OR}} \quad \int_0^{2\pi}\int_0^1 r^2\sin^2\theta \cdot r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 r^3\sin^2\theta\,dr\,d\theta$$

$$\frac{1}{4}\int_0^{2\pi} \sin^2\theta\,d\theta$$

$$\frac{1}{4}\int_0^{2\pi} \frac{1}{2}(1-\cos(2\theta))\,d\theta$$

$$\boxed{\pi/4 \text{ grams}}$$

4) If $\rho(x,y)$ is the mass density at point (x,y) on some thin flat object D (called a lamina), then

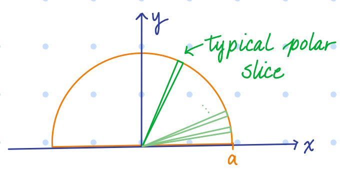
$$M_x = \text{moment of the lamina about the } x\text{-axis} = \iint_D y \cdot \rho(x,y)\, dA$$

$$M_y = \text{moment of the lamina about the } y\text{-axis} = \iint_D x \cdot \rho(x,y)\, dA$$

$$m = \text{mass} = \iint_D \rho(x,y)\, dA$$

$(\bar{x}, \bar{y})$ = coordinates of the center of mass, where $\bar{x} = \frac{M_y}{\text{mass}}$ and $\bar{y} = \frac{M_x}{\text{mass}}$

ex 4) A semicircular lamina has, at any point (x,y), density proportional to the distance from the center of the circle. Find the center of mass of the lamina.



For a point (r,θ), $\rho = kr$

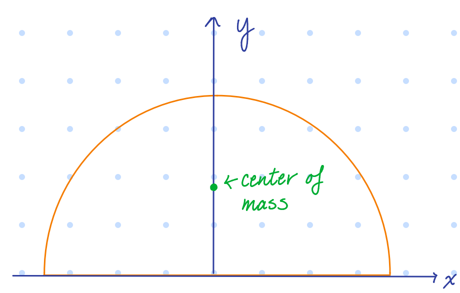
$$m = \iint_D \rho\, dA = \int_0^\pi \int_0^a kr \cdot r\, dr\, d\theta = \frac{1}{3} k\pi a^3$$

$$M_x = \iint_D y \cdot \rho\, dA = \int_0^\pi \int_0^a r\sin\theta \cdot kr \cdot r\, dr\, d\theta = \frac{1}{2} a^4$$

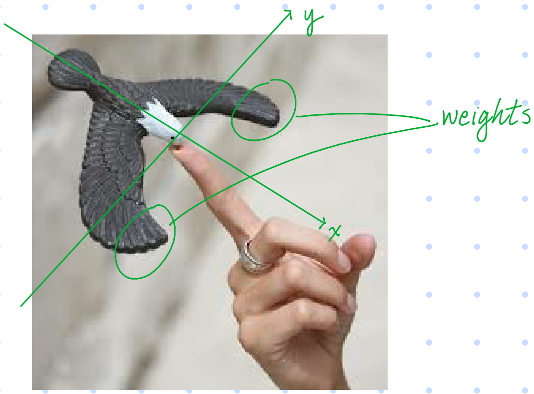
$$M_y = \iint_D x \cdot \rho\, dA$$

Notice ρ is symmetric across the y-axis and the lamina is also symmetric across the y-axis, but x becomes $-x$ on the other side of the y-axis. Based on these observations, $\iint_D x \cdot \rho\, dA = 0$

$$\text{center of mass} = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \boxed{\left(0, \frac{3a}{2\pi}\right)} \approx (0, 0.48a)$$



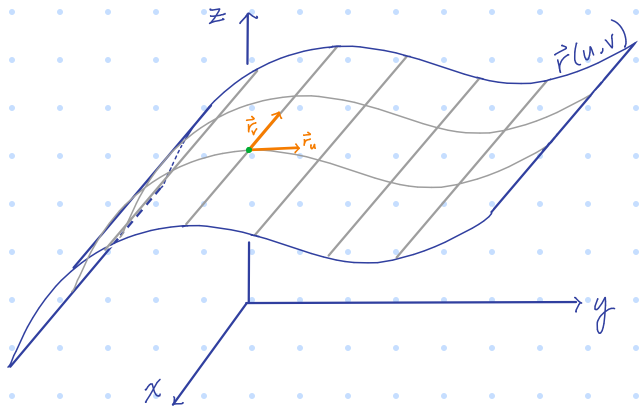
ex 5) Where is the center of mass of this balancing eagle toy?
Can you guess where there weights inside the toy?



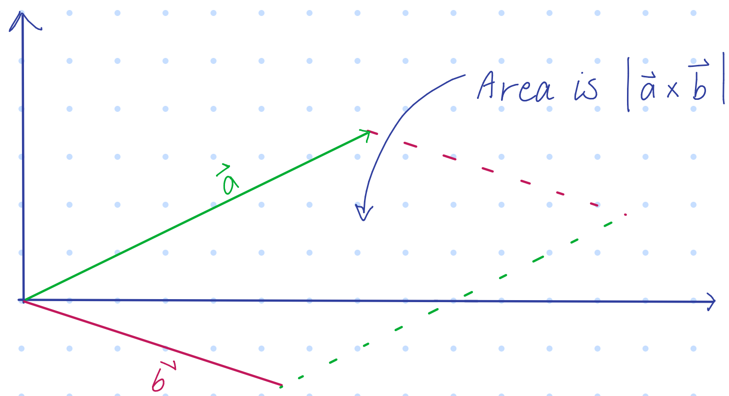
12.6 Surface area

First let's review two facts

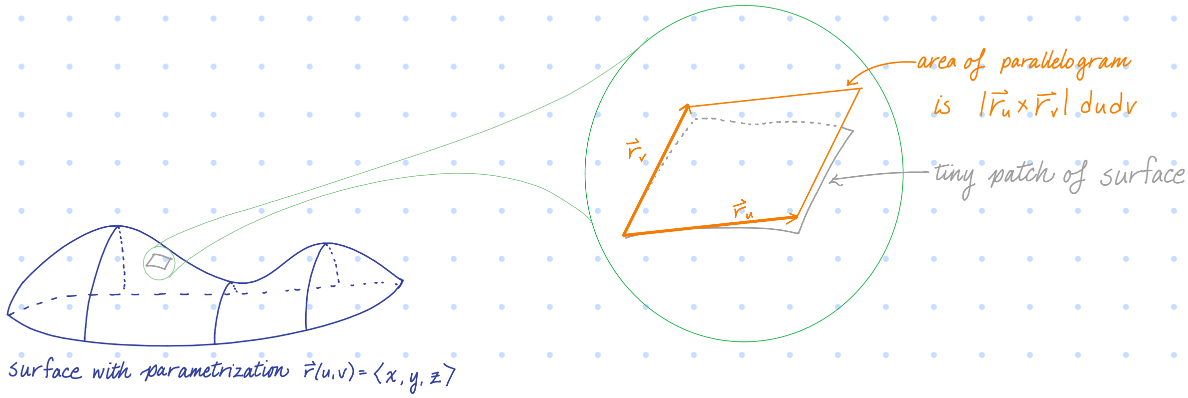
Fact 1: For vector-valued, two-parameter surfaces $\vec{r}(u,v)$, we know the partial derivatives $\vec{r}_u$ and $\vec{r}_v$ are vectors tangent to the surface (i.e. they lie in the tangent plane). [from notes on Sept 22, section 11.3]



Fact 2: $|\vec{a} \times \vec{b}|$ is a scalar that represents the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$
[from notes on Aug 31, section 9.4]



We can use these facts to see that a tiny patch of surface area on a parametrized surface $\vec{r}(u,v)$ is approximated by $|\vec{r}_u \times \vec{r}_v|\, du\, dv$



The surface area of a parametrized surface $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ with u,v in a region D is

$$\iint_D |\vec{r}_u \times \vec{r}_v|\, dA \qquad \left[\text{alternative notation: } \iint_D 1\, dS \quad \text{where } dS = |\vec{r}_u \times \vec{r}_v|\, dA\right]$$

ex 1) Find the surface area of a sphere of radius a.

Step 1: Parametrize surface

$\vec{r}(\theta,\phi) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi \rangle$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

Step 2: Compute area of tiny patch $|\vec{r}_\theta \times \vec{r}_\phi|$

$\vec{r}_\theta = \langle -a\sin\phi\sin\theta\ ,\ a\sin\phi\cos\theta\ ,\ 0 \rangle$

$\vec{r}_\phi = \langle a\cos\phi\cos\theta\ ,\ a\cos\phi\sin\theta\ ,\ -a\sin\phi \rangle$

$\vec{r}_\theta \times \vec{r}_\phi = \langle -a^2\sin^2\phi\cos\theta, -a^2\sin^2\phi\sin\theta\ , -a^2\sin\phi\cos\phi\sin^2\theta - a^2\sin\phi\cos\phi\cos^2\theta \rangle$

$= \langle -a^2\sin^2\phi\cos\theta, -a^2\sin^2\phi\sin\theta\ , -a^2\sin\phi\cos\phi \rangle$

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$
$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi}$$
$$= \sqrt{a^4\sin^2\phi(\sin^2\phi + \cos^2\phi)}$$
$$= a^2\sin\phi$$

Step 3: Integrate

$$\int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta = a^2\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta = \boxed{4\pi a^2}$$

Tuesday October 20

Reminders

- WebAssign 12.6
- Written HW section 12.6 and A1, A2

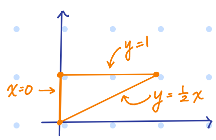
12.6 Surface area (cont)

Recap : The formula for surface area of a surface $\vec{r}(u,v)$ is $\iint_D 1\, dS = \iint_{u,v \text{ in } D} |\vec{r}_u \times \vec{r}_v|\, dA$

ex 2) Let S be the part of the surface $z = 1 + 3x + 2y^2$ above the triangle with vertices $(0,0)$, $(0,1)$, and $(2,1)$.

a) Write an integral that gives the volume under S and above $z = 0$.

Domain of integration



Volume under S

$$\boxed{\int_0^2 \int_{\frac{1}{2}x}^1 1 + 3x + 2y^2 \, dy\, dx}$$

b) Write an integral that gives the surface area of S and evaluate the integral.

$\vec{r}(s,t) = \langle s, t, 1 + 3s + 2t^2 \rangle \qquad 0 \le s \le 2, \quad \frac{1}{2}s < t < 1$

$\vec{r}_s = \langle 1, 0, 3 \rangle$

$\vec{r}_t = \langle 0, 1, 4t \rangle$

$\vec{r}_s \times \vec{r}_t = \langle -3, -4t, 1 \rangle$

$|\vec{r}_s \times \vec{r}_t| = \sqrt{9 + 16t^2 + 1}$

$= \sqrt{10 + 16t^2}$

surface area of $S = \int_0^2 \int_{\frac{1}{2}s}^1 \sqrt{10 + 16t^2}\, dt\, ds$

Hard to evaluate, so switch order of integration

$$\int_0^1 \int_0^{2t} \sqrt{10 + 16t^2}\, ds\, dt$$

$$\int_0^1 2t\sqrt{10 + 16t^2}\, dt \qquad \left(\begin{array}{l} u = 10 + 16t^2 \\ du = 32t\, dt \end{array} \right)$$

$$\frac{1}{16} \int_{10}^{26} u^{1/2}\, du$$

$$\frac{1}{16} \cdot \frac{2}{3} \left[u^{3/2} \right]_{10}^{26} = \boxed{\frac{26^{3/2} - 10^{3/2}}{24}} \approx 4.21$$

Optional formula for surface area of a surface $z = f(x,y)$

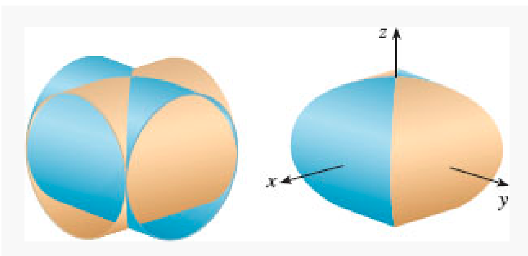
$$\text{Surface area} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

This formula is the result of choosing your parametrization to be $\langle x, y, f(x,y)\rangle$

It only works for functions of x and y, so it will not be useful for surfaces that fail the "vertical line test", like spheres.

ex 3) Find the surface area of the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$.

This is a WebAssign problem



See youtu.be/B-3CtvmSf3k for a better demo of this shape.

Strategy: Find surface area of just one of the four panels and multiply by 4 at the end.

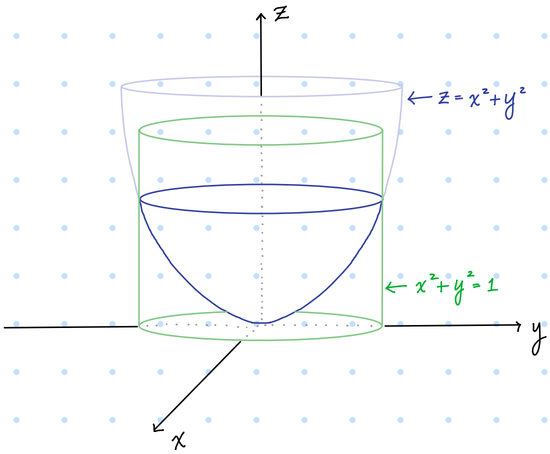
Step 1: parametrize one panel [Hint: The yellow panel on the positive y side is the part of surface $y = \sqrt{1-z^2}$ inside $x^2 + z^2 = 1$]

Step 2: compute $|\vec{r}_u \times \vec{r}_v|$

Step 3: integrate $\displaystyle\iint_{u,v \text{ in } D} |\vec{r}_u \times \vec{r}_v|\, dA$

Check-in 14

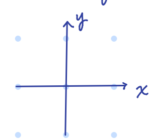
Find the volume under the paraboloid $z = x^2+y^2$ within the cylinder $x^2+y^2 \le 1$, $z \ge 0$.



Optional guidance

a) Identify what part of the diagram we are interested in or draw your own.

b) Draw the projection of the object onto the xy-plane.



c) Use the projection to write bounds for a double integral. You may use rectangular ($\iint f\,dy\,dx$ or $\iint f\,dx\,dy$) or polar ($\iint f\,r\,dr\,d\theta$)

d) Decide what your integrand f should be

e) Integrate. Make sure your final answer is a number.

Wednesday October 21

Reminders

- Compile HW 9 for André
- [Thurs] Study for Check-in 15 - practice surface area

12.6 Surface area (cont)

Let's explore what it means to integrate the function 1.

- For a single integral: $\boxed{\int_a^b 1\, dx}$

  This integral returns the value $b-a$, which is the length of the interval $[a,b]$

  [from single-variable calculus]

- For a double integral: $\boxed{\iint_D 1\, dA}$

  This integral returns the area of the region D

  [from section 12.3]

- For a triple integral: $\boxed{\iiint_E 1\, dV}$

  This integral returns the area of the region D

  [from section 12.7 - the future, ooh]

Key concept

The integral of the function 1 produces the "size" of the domain of integration

The "size" must be appropriately interpreted as length, area, volume, etc.

ex 4) Compute the surface area of the part of $z = -3x + 7y$ above the region D, where D is a blob of area 9.

Parametrize surface: $\vec{r}(p,q) = \langle p, q, -3p + 7q \rangle$

Compute $|\vec{r}_p \times \vec{r}_q|$: $\vec{r}_p = \langle 1, 0, -3 \rangle$

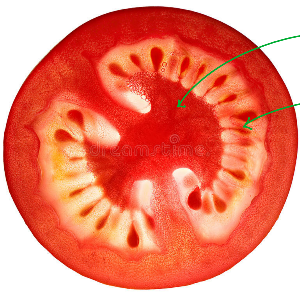
$\vec{r}_q = \langle 0, 1, 7 \rangle$

$|\vec{r}_p \times \vec{r}_q| = \sqrt{3^2 + (-7)^2 + 1^2} = \sqrt{59}$

Integrate over D: $\iint_D \sqrt{59}\, dA = \sqrt{59} \underbrace{\iint_D dA}_{\text{area of } D} = \boxed{9\sqrt{59}}$

12.7 Triple integrals

For a lamina D, if the density at (x,y) is given by $\rho(x,y)$,
then $\iint_D \rho(x,y)\,dA$ is the mass of the lamina.



more dense here

less dense here

$\rho(x,y)$ gives density of tomato mass at (x,y)

$\iint_{\text{slice}} \rho(x,y)\,dA$ gives mass of whole slice

But not every object is thin and flat! How do we find the mass of a 3D object?
By cutting up the solid into thin slices!

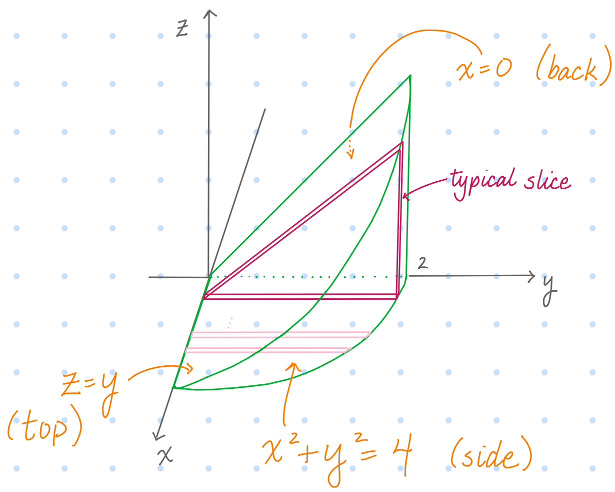
$$\text{total mass of tomato} = \int_a^b \underbrace{\left(\text{mass of each slice}\right)}_{\text{This is already a double integral}} dx$$

To compute a triple integral over a solid V, slice it like so:

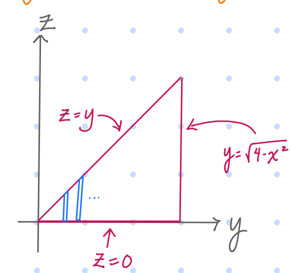
$$\iiint_E f(x,y,z)\,dV = \int_e^g \underbrace{\int_{c(x)}^{d(x)} \underbrace{\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz}_{\substack{z \text{ is the variable,}\\ x \text{ and } y \text{ are fixed}}}\,dy}_{\substack{y \text{ is the variable,}\\ x \text{ is fixed, } z \text{ is gone}}}\,dx$$

ex 1) Evaluate $\iiint_E 2x\,dV$ where $E = \{(x,y,z) \mid 0 \le y \le 2,\ 0 \le x \le \sqrt{4-y^2},\ 0 \le z \le y\}$

Draw E, the domain of integration



Draw typical slice of E



Use typical slice to write two inner bounds.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^y 2x \; dz\,dy\,dx = \int_0^2 \int_0^{\sqrt{4-x^2}} 2xy \; dy\,dx$$

$$= \int_0^2 2x \cdot \tfrac{1}{2} \left[y^2 \right]_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$= \int_0^2 x(4-x^2)\,dx$$

$$= \int_0^2 4x - x^3\,dx$$

$$= \left[2x^2 - \tfrac{1}{4}x^4 \right]_{x=0}^{x=2}$$

$$= 8 - 4$$

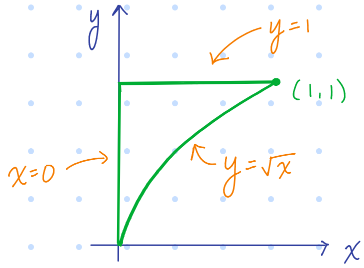
$$= \boxed{4}$$

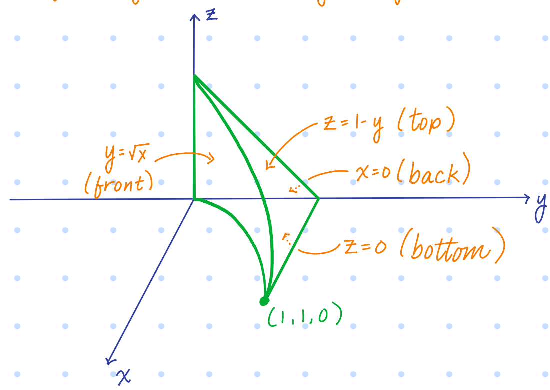
ex 2) Rewrite following integral as an equivalent iterated integral in the five other orders.

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$$

Very popular exam question

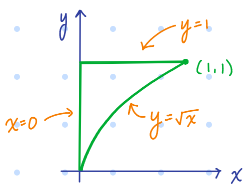
Draw domain of integration by drawing region given by two outer integrals first





(1) dz dx dy (swap outer two variables of original integral)

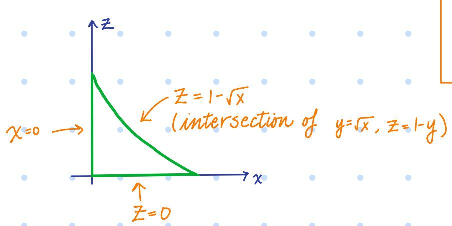
Draw xy-projection



$$\boxed{\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\, dz\, dx\, dy}$$

(2) dy dz dx (swap inner two variables of original integral)

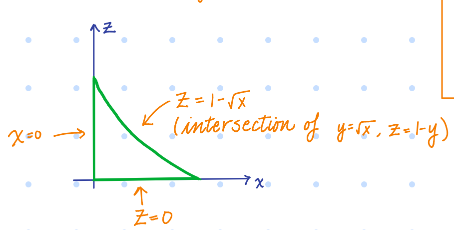
Draw zx-projection



$$\boxed{\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dz\, dx}$$

(3) dy dx dz (swap outer two variables of (2))

Draw xz-projection

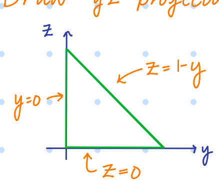


$$\boxed{\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dx\, dz}$$

Computational note: Switching the inner two or outer two variables is easier than switching two non-adjacent variables

(4) dx dy dz and (5) dx dz dy

Draw yz-projection



$$\boxed{\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\, dx\, dy\, dz}$$

$$\boxed{\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\, dx\, dz\, dy}$$

Friday October 23

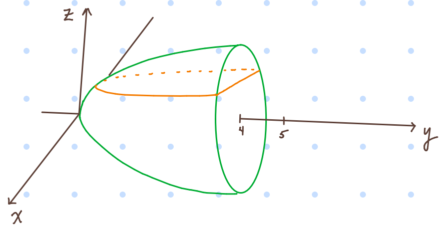
Reminders

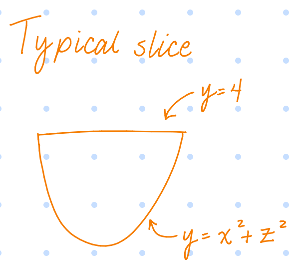
- All week 9 WebAssign due Sunday before midnight
- Written HW 10, section 12.7
- Study for Quiz 5

Recap: For a triple integral $\iiint f(x,y,z)\, dz\, dy\, dx$, we can write numerical bounds for the outermost integral and progress inward by adding further restrictions for the inner integrals

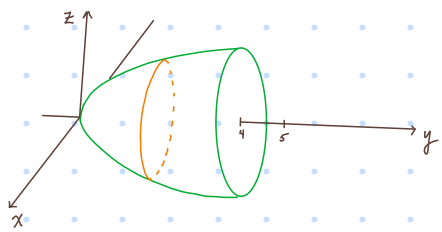
ex 3) Let E be the solid bounded by $y = x^2 + z^2$ and $y = 4$.

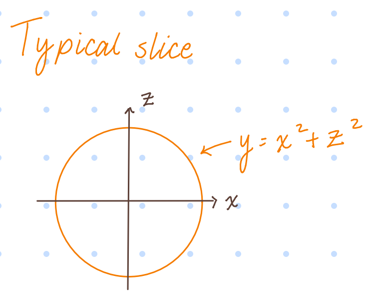
a.) Draw E and a typical slice of E for a fixed z.





b.) Draw E and a typical slice of E for a fixed y.





c) Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$

$$\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\; dz\, dx\, dy$$

This looks well-suited to polar coordinates

$x = r\cos\theta,\ z = r\sin\theta$

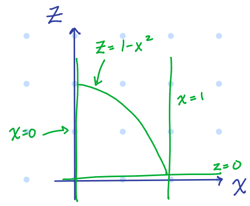
$$\int_0^4 \left(\int_0^{2\pi} \int_0^{\sqrt{y}} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \cdot r\, dr\, d\theta \right) dy$$

$$\int_0^4 \left(\int_0^{2\pi} \int_0^{\sqrt{y}} r^2\, dr\, d\theta \right) dy = \dots \boxed{\frac{128\pi}{15}}$$

ex 4) Rewrite $\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\ dy\,dz\,dx$ in the five other orders of integration

(1) dy dx dz

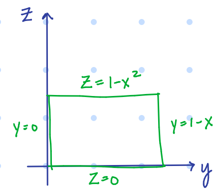
Flip x, z in original integral



$$\int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\ dy\,dx\,dz$$

(2) dz dy dx

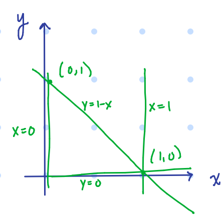
Flip y, z in original integral



$$\int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\ dz\,dy\,dx$$

(3) dz dx dy

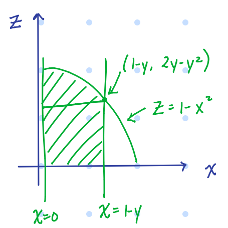
Flip x, y in (2)



$$\int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\ dz\,dx\,dy$$

(4) dx dz dy

Flip x, z in (3)

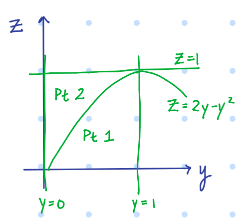


$$\int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\ dx\,dz\,dy + \int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} f(x,y,z)\ dx\,dz\,dy$$

$$z = 1-(1-y)^2 = 1-(1-2y+y^2) = 2y-y^2$$

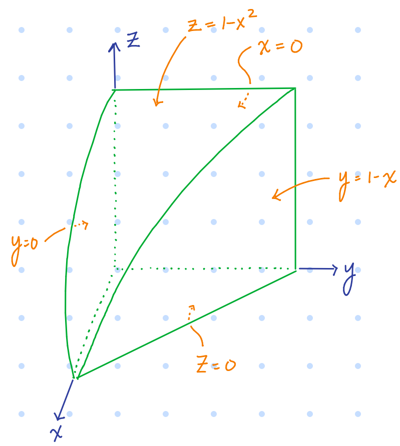
(5) dx dy dz

Flip y, z in (4)



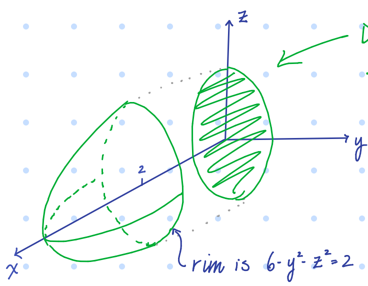
$$\int_0^1\int_{1+\sqrt{1-z}}^1\int_0^{1-y} f(x,y,z)\ dx\,dy\,dz + \int_0^1\int_0^{1+\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\ dx\,dy\,dz$$

3D graph

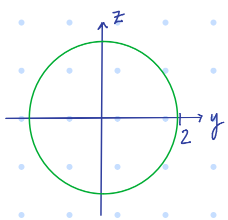


Check-in 15

Compute the surface area of the portion of $x = 6 - y^2 - z^2$ where $x \geq 2$.



Domain of integration



$$SA = \iint 1\, dS$$

$$= \iint \sqrt{1 + f_y^2 + f_z^2}\; dA \qquad (\text{special case of } x = f(y,z))$$

$$= \iint \sqrt{1 + (-2y)^2 + (-2z)^2}\; dA$$

$$= \iint \sqrt{1 + 4(y^2 + z^2)}\; dA$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2} \cdot r\,dr\,d\theta \qquad (y = r\cos\theta,\; z = r\sin\theta)$$

$$= 2\pi \left[(1+4r^2)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{8} \right]_{r=0}^{r=2}$$

$$= \frac{\pi}{6}\left[17^{3/2} - 1 \right]$$

Monday October 26

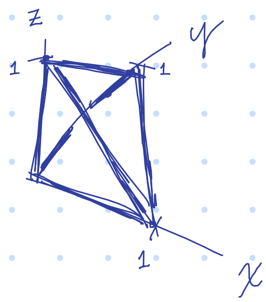
Class was cancelled for snow, we had a review day for Quiz 5

Reminders
- Study for Quiz 5

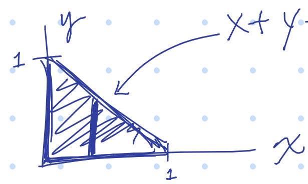
Textbook section 12.7 prob 40

Find mass, center of mass of E w/ density ρ

E = tetrahedron bounded by $x=0,\ y=0,\ z=0,\ x+y+z=1$, $\rho(x,y,z)=y$



xy projection



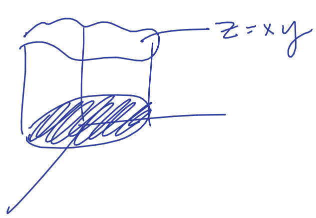
$x+y+0=1$

$y=1-x$

$$m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \; dz\, dy\, dx$$

Name: ____

1. Determine whether each statement is TRUE or FALSE
   (a) $\int_0^1 \int_0^x \sqrt{x+y^2}\, dy\, dx = \int_0^x \int_0^1 \sqrt{x+y^2}\, dx\, dy.$
   (b) If D is the unit disk centered at the origin, then $\iint_D f(x,y)\, dA = \int_{-1}^1 \int_{-1}^1 f(x,y)\, dy\, dx.$
   (c) If D is the unit disk centered at the origin, then $\iint_D f(x,y)\, dA = \int_0^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta)\, dr\, d\theta.$
   (d) The double integral $\iint_D dA$ is always positive.
   (e) The triple integral $\iiint_E dV$ is always positive.
2. Evaluate the integral.
   (a) $\int_0^1 \int_0^x \cos(x^2)\, dy\, dx$
   (b) $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin x\, dz\, dy\, dx$
   (c) $\iint_D \frac{y}{1+x^2}\, dA$ where $D = \{(x,y) \mid 0 \le y \le 1,\ y^2 \le x \le y+2\}$
   (d) $\iiint_E xy\, dV$ where $E = \{(x,y,z) \mid 0 \le x \le 3,\ 0 \le y \le x,\ 0 \le z \le x+y\}$
   (e) $\iint_D (x^2+y^2)^{3/2}\, dA$ where D is the region in the first quadrant bounded by the lines $y=0$, $y=\sqrt{3}x$ and $x^2+y^2=9$
   (f) $\iiint_E yz\, dV$ where E lies above the plane $z=0$, below the plane $z=y$ and inside the cylinder $x^2+y^2=4$.
3. Find the volume of the solid tetrahedron with vertices $(0,0,0)$, $(0,0,1)$, $(0,2,0)$, and $(2,2,0)$.
4. Find the volume of the wedge cut from the cylinder $x^2+9y^2=a^2$ by the planes $z=0$ and $z=mx$.
5. Sketch the solid whose volume is given by $\int_0^2 \int_0^{2-y} \int_0^{2-x-y} dz\, dx\, dy$
6. Find the surface area.
   (a) The part of the plane $3x+2y+z=6$ that lies in the first octant.
   (b) The part of the surface $z=xy$ that lies within the cylinder $x^2+y^2=1$
7. A lamina D lies on the xy-axes bounded by the parabola $x=1-y^2$ and the coordinate axes in the first quadrant with density function $\rho(x,y)=y$. Find the center of mass of D.



$SA = \iint \sqrt{1+z_x^2+z_y^2}\; dA$

$= \iint \sqrt{1+y^2+x^2}\; dA$

$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\,(r)\,dr\,d\theta$

$1+r^2 = u$
$2r\,dr = du$

$= \frac{1}{2}\int_0^{2\pi}\int_1^2 u^{1/2}\, du\, d\theta$

8. A lamina occupies a circular disk D whose center lines on the line $y = 2$ and whose density is given by the function $\rho(x, y) = x^2 + y^2$. Determine whether the following statements are true, false, or if not enough information is given.

   (a) If the center of D is (x_0, y_0) and the center of mass of the lamina is $(\overline{x}, \overline{y})$, then $\overline{x} > x_0$.

   (b) If the center of D is (x_0, y_0) and the center of mass of the lamina is $(\overline{x}, \overline{y})$, then $\overline{y} > y_0$.

9. Rewrite $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x, y, z)\, dz\, dy\, dx$ as an iterated integral in ~~the order $dx\, dy\, dz$~~. *all possible orders*

10. Decide, without calculation, whether the integrals are positive, negative, or zero. Let D be the region inside the unit circle centered at the origin, let R be the right half of D, and let B be the bottom half of D.

    (a) $\iint_D dA$

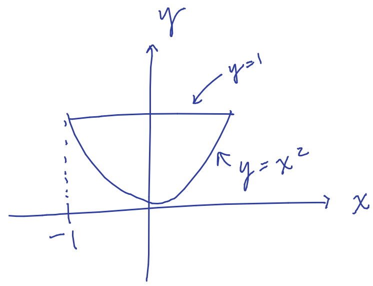
    (b) $\iint_R 5x\, dA$

    (c) $\iint_D 5x\, dA$

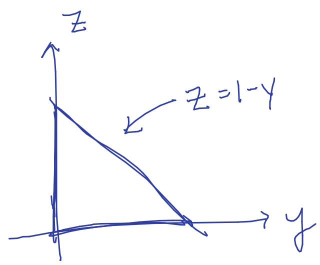
    (d) $\iint_B y^3 + y^5\, dA$

    (e) $\iint_D \sin y\, dA$

    (f) $\iint_D xy^2\, dA$

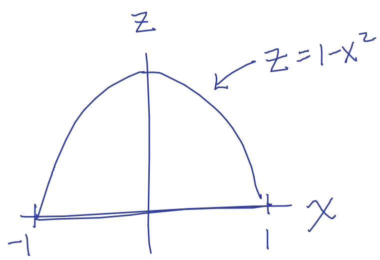


$$\int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f\, dx\, dy\, dz$$



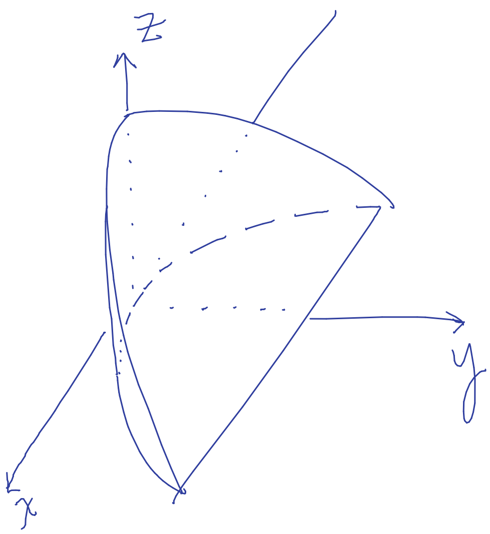
$$\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{1-y} f\, dz\, dx\, dy$$

$$\int_0^1\int_0^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} f\, dx\, dz\, dy$$



$$\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} f\, dy\, dx\, dz$$

$$\int_{-1}^1\int_0^{1-x^2}\int_{x^2}^{1-z} f\, dy\, dz\, dx$$



Answers

(1) *F, F, F, T, T*

(2) *(a)* $(1/2)sin(1)$ *(b)* $2/3$ *(c)* $(1/4)\ln 2$ *(d)* $81/2$ *(e)* $81\pi/5$ *(f)* $64/15$

(3) $2/3$

(4) $2ma^3/9$

(5) *tetrahedron with corners* $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, $(0,0,2)$

(6) *(a)* $3\sqrt{14}$ *(b)* $(2\pi/3)(2\sqrt{2}-1)$

(7) $(1/3, 8/15)$

(8) *C, A*

(9) $\displaystyle\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z,)\,dx\,dy\,dz$

(10) *(a) pos (b) pos (c) zero (d) neg (e) zero (f) zero*

University of Colorado Boulder

Math 2400, Midterm 3

Spring 2017

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

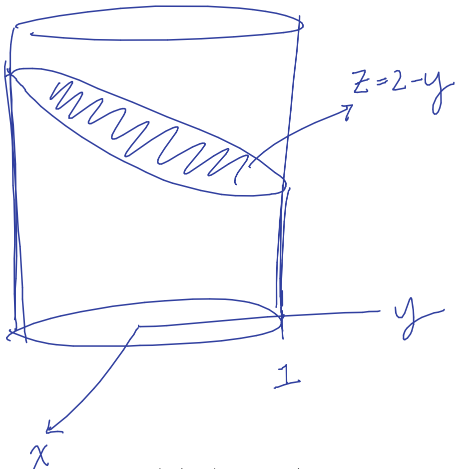
SECTION #: ______________________________

Question	Points	Score
1	13	
2	16	
3	8	
4	8	
5	15	
6	8	
7	16	
8	16	
Total:	100	

- No calculators, cell phones, or other electronic devices may be used at any time during the exam.
- Show all of your reasoning and work for full credit, unless indicated otherwise. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like $100/7$ or expressions like $\ln(3)/2$ as is.
- For multiple choice questions, circle the correct answer.
- Notation: following the book, we use boldface to denote vectors, e.g., $\mathbf{a}, \mathbf{b}$ are vectors.

8. (16 points) Let S be the surface of the solid obtained by taking a section of the cylinder $x^2 + y^2 = 1$ between the planes $z = 2 - y$ and $z = 0$.

(a) (6 pts.) The upper face of S, the part lying in the plane $z = 2 - y$, may be parametrized by $\mathbf{r}(x, y) = \langle x, y, 2 - y \rangle$, where $(x, y) \in \{(x, y) : x^2 + y^2 \leq 1\}$. Compute the surface area of that portion of the surface S.



$$\iint \sqrt{1 + z_x^2 + z_y^2}\, dA$$

$$\boxed{\sqrt{2}\,\pi}$$

$$\iint \sqrt{1 + 0 + 1}\, dA$$

$$\sqrt{2} \underbrace{\iint dA}_{\text{Area of circle}} = \sqrt{2} \cdot \pi\, 1^2$$

(b) (6 pts.) The portion of the surface S lying on the cylinder $x^2 + y^2 = 1$ may be parametrized by $\mathbf{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z \rangle$. Find the bounds for θ and z and then calculate the surface area of that portion of S.

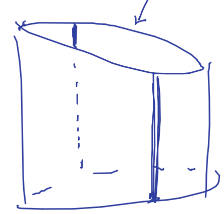
$$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$$

$$\vec{r}_z = \langle 0, 0, 1 \rangle$$

$$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2 + (0)^2}$$

$$= 1$$

(top rim)
$z = 2 - y$
$z = 2 - \sin\theta$



$$\boxed{\int_0^{2\pi} \int_0^{2 - \sin\theta} 1\, dz\, d\theta}$$

(c) (4 pts.) What is the total surface area of S?

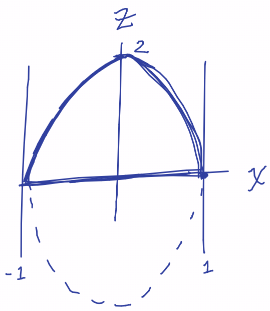
$$\underset{\text{top}}{(a)} + \underset{\text{side}}{(b)} + \underset{\text{bottom}}{\pi}$$

6. Consider the triple integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{4-2x^2}}\int_{-\sqrt{4-2x^2-z^2}}^{\sqrt{4-2x^2-z^2}} x\,dy\,dz\,dx$$

Without sketching the three-dimensional region of integration, use your understanding of changing the order of integration for double integrals to do the following.

(a) Change the order of integration to $dy\,dx\,dz$.



$z=\sqrt{4-2x^2}$

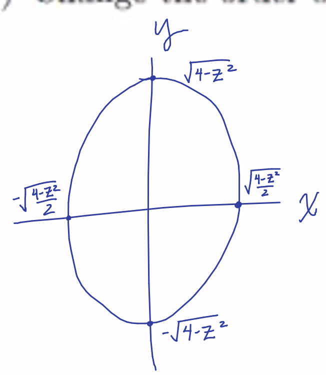
$z^2+2x^2=4$

ellipse $(\pm 1, 0)$ $(0, \pm 2)$

$x=\pm\sqrt{\frac{4-z^2}{2}}$

$$\int_0^2\int_{-\sqrt{\frac{4-z^2}{2}}}^{\sqrt{\frac{4-z^2}{2}}}\int_{-\sqrt{4-2x^2-z^2}}^{\sqrt{4-2x^2-z^2}} x\,dy\,dx\,dz$$

(b) Change the order of integration to $dx\,dy\,dz$.



$y^2=4-2x^2-z^2$

$y^2+2x^2=4-z^2$

ellipse $(0, \pm\sqrt{4-z^2})$ $(\pm\sqrt{\frac{4-z^2}{2}}, 0)$

$$\int_0^2\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{-\sqrt{\frac{4-z^2-y^2}{2}}}^{\sqrt{\frac{4-z^2-y^2}{2}}} x\,dx\,dy\,dz$$

(c) Evaluate the triple integral in the easiest order.

$$\int_0^2\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{-\sqrt{\frac{4-z^2-y^2}{2}}}^{\sqrt{\frac{4-z^2-y^2}{2}}} x\,dx\,dy\,dz = \frac{1}{2}\int_0^2\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\Big[x^2\Big]_{x=}^{x=} dy\,dz$$

$$= \frac{1}{2}\int_0^2\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} 0\,dy\,dz$$

$$= \boxed{0}$$

Mon Tue review Wed spherical + cyl practice

Corrections - explain whyyyyy...

Polar notes

dA in rect is $dy\,dx$ or

in polar is $r\,dr\,d\theta$

dV in rect is $dz\,dy\,dx$

cylindrical is $r\,dr\,d\theta\,dz$

spherical is $\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

Tuesday October 27

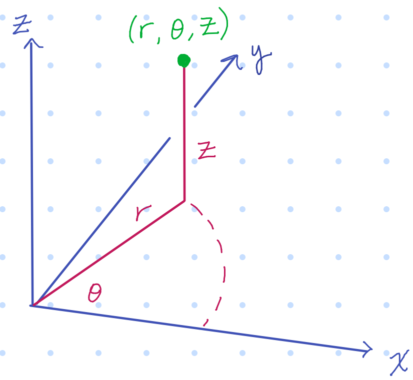
Reminders

- [Important] Do Quiz 5 between 7 and 10 pm (a timer is recommended)

12.8 Triple integrals in cylindrical and spherical

Refresher on cylindrical and spherical coordinates

Cylindrical coordinates

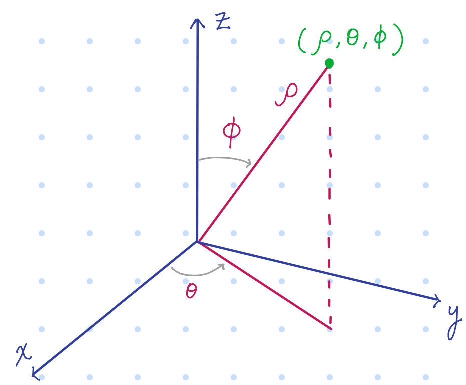


$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

Spherical coordinates



$\rho^2 = x^2 + y^2 + z^2$

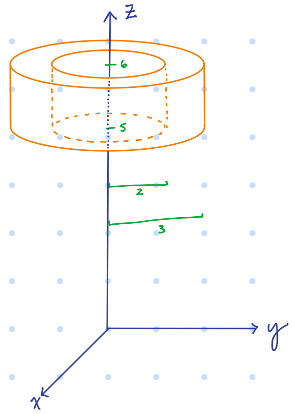
$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

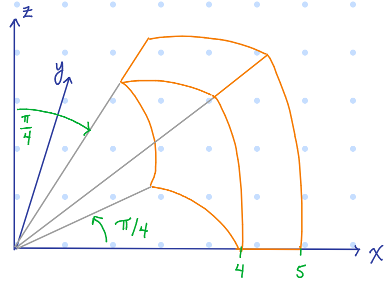
$z = \rho\cos\phi$

Sketch each solid

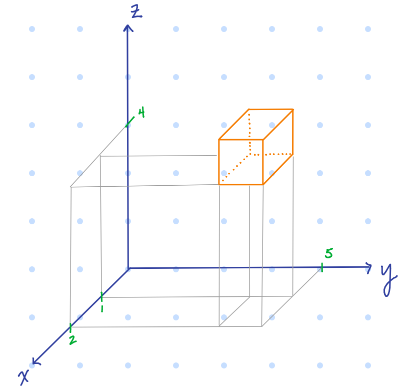
(a) $2 \le r \le 3$
$0 \le \theta \le 2\pi$
$5 \le z \le 6$



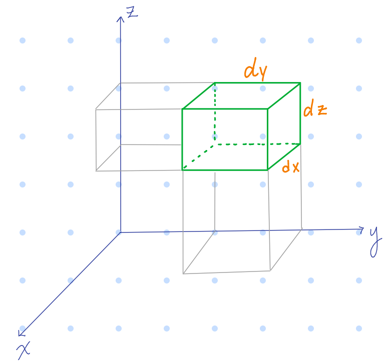
(b) $4 \le \rho \le 5$
$0 \le \theta \le \pi/4$
$\pi/4 \le \phi \le \pi/2$



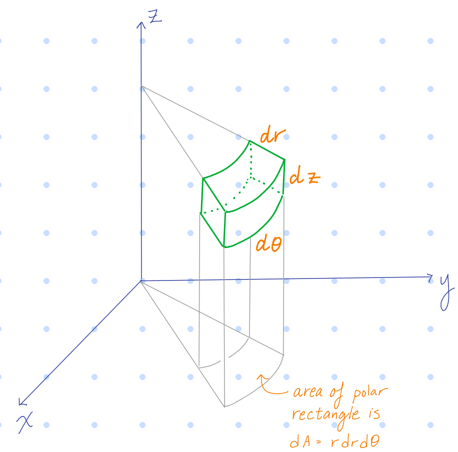
(c) $1 \le x \le 2$
$4 \le y \le 5$
$4 \le z \le 5$



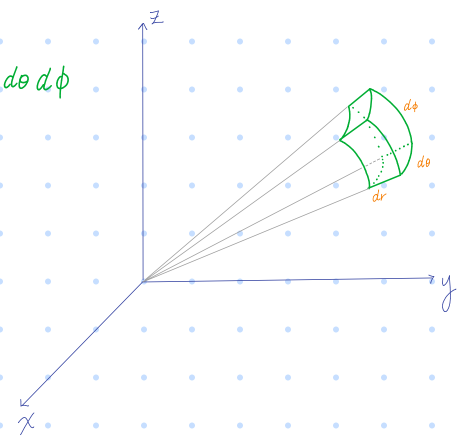
The volume element dV in rectangular coordinates is $dz\,dy\,dx$
The tiny solid formed by moving $x, y,$ and z by a small amount is a rectangular block, so the volume is $dz\,dy\,dx$



The volume element dV in cylindrical coordinates is $r\,dr\,d\theta\,dz$
The tiny solid formed by moving $r, \theta,$ and z by a small amount is a polar rectangle stretched upward by dz



The volume element dV in spherical coordinates is $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$
The ny solid formed by moving $\rho, \theta,$ and ϕ by a small amount is sh ed like um... I don't know.

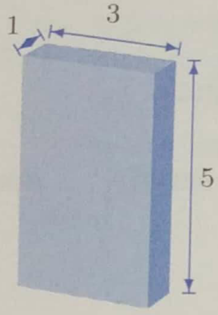


The triple integral $\iiint_E f(x,y,z)\,dV$ over solid E is

- (in rectangular) $\displaystyle\iiint_{E \text{ in rectangular}} f(x,y,z)\;dz\,dy\,dx$
- (in cylindrical) $\displaystyle\iiint_{E \text{ in cylindrical}} f(r\cos\theta, r\sin\theta, z)\;r\,dr\,d\theta\,dz$
  (often $r\,dz\,dr\,d\theta$ is useful)
- (in spherical) $\displaystyle\iiint_{E \text{ in spherical}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\;\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

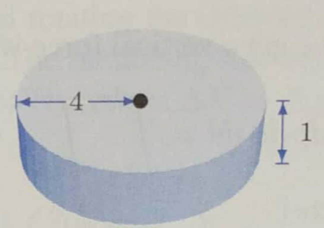
For Exercises 12–18, choose coordinates and set up a triple integral, including limits of integration, for a density function f over the region.

12.



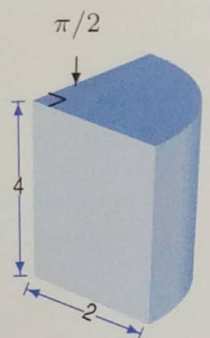
12) $\int_0^5 \int_0^3 \int_0^1 f(x,y,z)\, dx\,dy\,dz$

13.



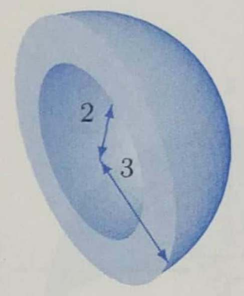
13) $\int_0^{2\pi} \int_0^4 \int_0^1 f(r\cos\theta, r\sin\theta, z)\, r\, dz\,dr\,d\theta$

14.



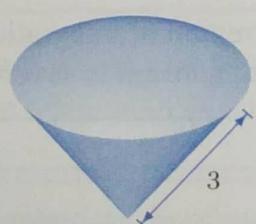
14) $\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^4 f(r\cos\theta, r\sin\theta, z)\, r\, dz\,dr\,d\theta$

15.



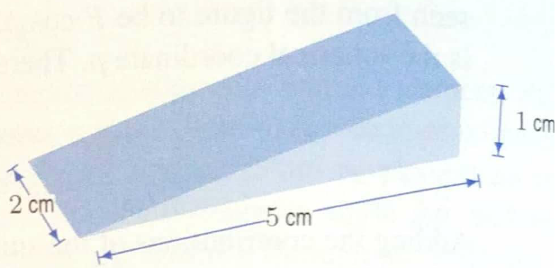
15) $\int_0^{\pi} \int_0^{\pi} \int_2^3 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\,d\phi\,d\theta$

16. A piece of a sphere; angle at the center is $\pi/3$.



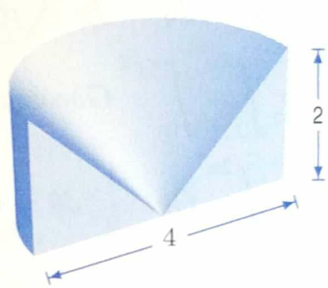
16) $\int_0^{2\pi} \int_0^{\pi/6} \int_0^3 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\,d\phi\,d\theta$

17.



17) $\int_0^2 \int_0^5 \int_0^{\frac{1}{5}y} f(x,y,z)\, dz\,dy\,dx$

18.

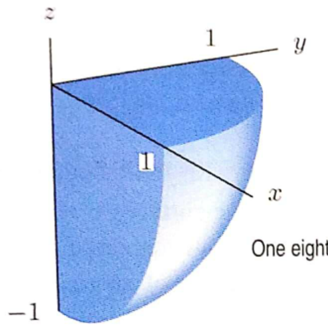


18) $\int_0^{\pi} \int_0^2 \int_0^r f(r\cos\theta, r\sin\theta, z)\, r\, dz\,dr\,d\theta$

For the regions W shown in Problems 30–32, write the limits of integration for $\int_W dV$ in the following coordinates:

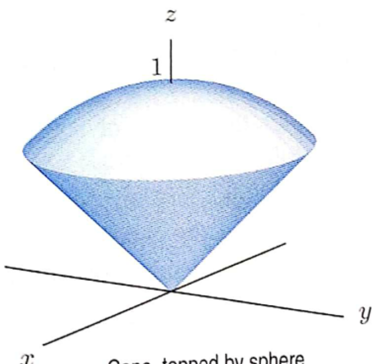
(a) Cartesian **(b)** Cylindrical **(c)** Spherical

30.



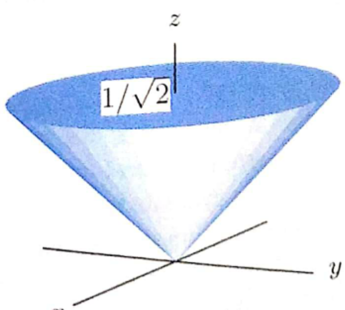
One eighth sphere

31.



Cone, topped by sphere of radius 1 centered at origin, 90° at vertex

32.



Cone, flat on top, $\pi/2$ at vertex

30)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{0} f(x,y,z)\, dz\, dy\, dx$$

$$\int_0^{\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{0} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

$$\int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_0^1 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

31)

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx$$

$$\int_0^{2\pi} \int_0^{1/2} \int_r^{\sqrt{1-r^2}} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

$$\int_0^{\pi/4} \int_0^{2\pi} \int_0^1 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

32)

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\frac{1}{\sqrt{2}}} f(x,y,z)\, dz\, dy\, dx$$

$$\int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_r^{\frac{1}{\sqrt{2}}} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

$$\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}\sec\phi} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

Wednesday October 28

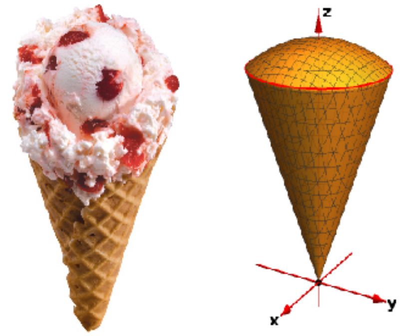
Reminders
- Compile HW 10 for André
- Submit Quiz 5 corrections by 10 pm - why, not how

12.8 Triple integrals in cylindrical + spherical (cont)

Recap: The triple integral $\iiint_E f(x,y,z)\,dV$ over solid E is

- (in rectangular) $\displaystyle\iiint_{E \text{ in rectangular}} f(x,y,z)\,dz\,dy\,dx$
- (in cylindrical) $\displaystyle\iiint_{E \text{ in cylindrical}} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$
  (often $r\,dz\,dr\,d\theta$ is useful)
- (in spherical) $\displaystyle\iiint_{E \text{ in spherical}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

Ex 1 Let E be the solid inside the cone $z = 4\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=9$. If the density at any point in E is proportional to z, find the mass of E. $\left[\textbf{Ans: } \frac{81k\pi}{68}\right]$

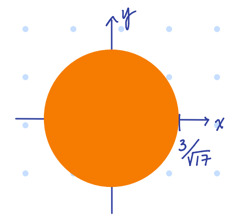


http://factsofun.wordpress.com/

In rectangular: $\rho(x,y,z) = kz$

$$m = \int_{-3/\sqrt{17}}^{3/\sqrt{17}} \int_{-\sqrt{\frac{9}{17}-x^2}}^{\sqrt{\frac{9}{17}-x^2}} \int_{4\sqrt{x^2+y^2}}^{\sqrt{9-x^2-y^2}} kz \; dz\,dy\,dx$$

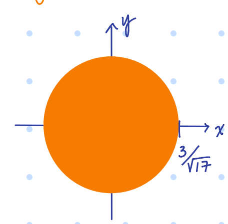
xy projection



In cylindrical: $\rho(r,\theta,z) = kz$

$$m = \int_0^{2\pi} \int_0^{3/\sqrt{17}} \int_{4r}^{\sqrt{9-r^2}} kz \cdot r\,dz\,dr\,d\theta$$

xy projection

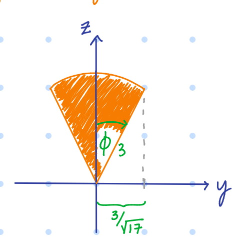


changed name of density to σ

In spherical: $\sigma(\rho,\theta,\phi) = k\rho\cos\phi$

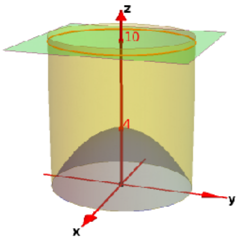
$$m = \int_0^{\arcsin\left(\frac{1}{\sqrt{17}}\right)} \int_0^{2\pi} \int_0^{3} k\rho\sin\phi \cdot \rho^2\sin\phi \; d\rho\,d\theta\,d\phi$$

yz projection



$\sin\phi = \frac{1}{\sqrt{17}}$

Ex 2 A solid E lies within the cylinder $x^2+y^2=4$, above the paraboloid $z=4-x^2-y^2$, and below the plane $z=10$. If the density of E at any point is proportional to its distance from the axis of the cylinder.



(a) Find the mass of E. [**Ans:** $\frac{224k\pi}{5}$] Practice for Quiz 6

(b) Find the volume of the solid E. [**Ans:** 32π] Practice for Quiz 6

(c) Find the area of the bottom surface of E, i.e., the paraboloid within the cylinder. [**Ans:** $\frac{1}{6}(17\sqrt{17}-1)\pi$] Practice for Quiz 5

(b) $$V=\iiint_E 1\,dV=\int_0^{2\pi}\int_0^2\int_{4-r^2}^{10} r\,dz\,dr\,d\theta$$

(a) $\rho=k\sqrt{x^2+y^2}$,

$$m=\iiint k\sqrt{x^2+y^2}\,dV$$

$$=\int_0^{2\pi}\int_0^2\int_{4-r^2}^{10} kr\cdot r\,dz\,dr\,d\theta$$

Ex 3 Use cylindrical coordinates to evaluate $\iiint_E x\,\mathrm{d}V$, where E is enclosed by the planes $z=0$, $z=x+y+10$, and by the cylinders $x^2+y^2=16$, $x^2+y^2=36$. [**Ans:** 260π]

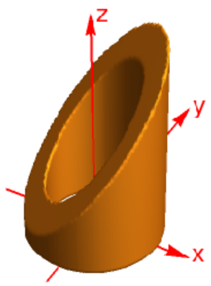
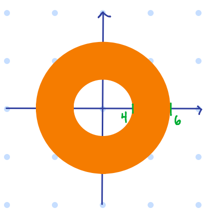


Fig for **Ex 3**

xy projection



$$\iiint_E x\,dV = \int_0^{2\pi}\int_4^6\int_0^{r\cos\theta + r\sin\theta + 10} r\cos\theta \cdot r\,dz\,dr\,d\theta$$

Ex 4 Evaluate the iterated integral by changing to cylindrical coordinates:

$$\int_{-5}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{25-x^2-y^2}\sqrt{x^2+y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \qquad \left[\textbf{Ans: } \frac{1250\pi}{3}\right]$$

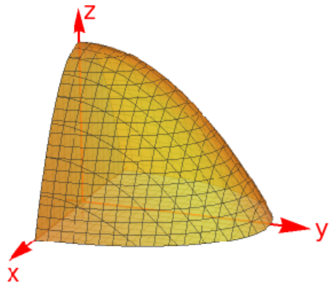
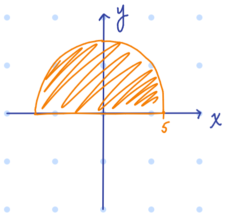


Fig for **Ex 4**

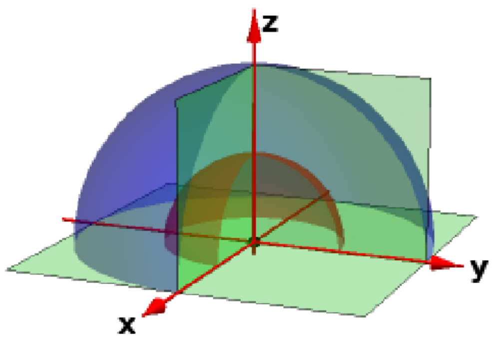
xy projection



$-5 \le x \le 5$
$0 \le y \le \sqrt{25-x^2}$

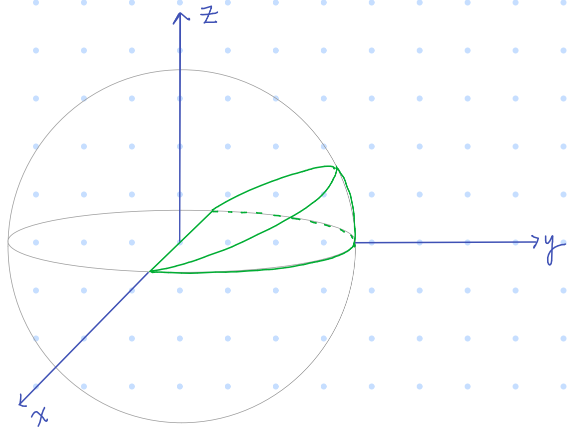
$$\int_0^{\pi}\int_0^{5}\int_0^{5-r^2} r^2 \, dz\, dr\, d\theta$$

Ex 5 Evaluate the integral $\iiint\limits_E \ln\left(1+(x^2+y^2+z^2)^{\frac{3}{2}}\right) \mathrm{d}V$, where the E is the solid formed by between the two spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$ in the second to the 4th octants.
[**Ans:** $\pi\,[14\ln(28)-13-\ln 2]$]



$$\int_1^3 \int_{\pi/2}^{2\pi} \int_0^{\pi/2} \ln(1+\rho^3)\cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

ex 6) Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.



$$\int_{\pi/3}^{\pi/2} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \; d\rho \, d\theta \, d\phi$$

$$\boxed{\frac{a^3\pi}{6}}$$

Friday October 30

Q6 MWS: 4 questions
resubmissions allowed.

Cumulative quiz study materials

Reminders

- All Week 10 WebAssign due Sunday before midnight
- Review: what is the gradient of a function

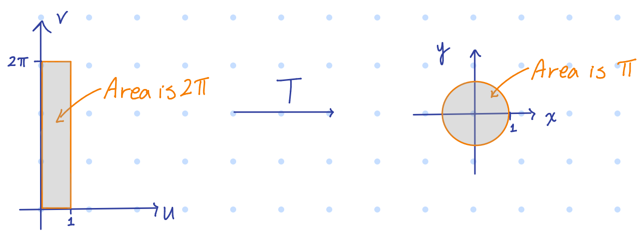
12.9 Change of variables

A *tranformation* T is a function whose domain and range are both $\mathbb{R}^n$

Examples

- $\vec{T}(u,v) = \langle u\cos v, u\sin v\rangle$, $0 \le u \le 1$, $0 \le v \le 2\pi$

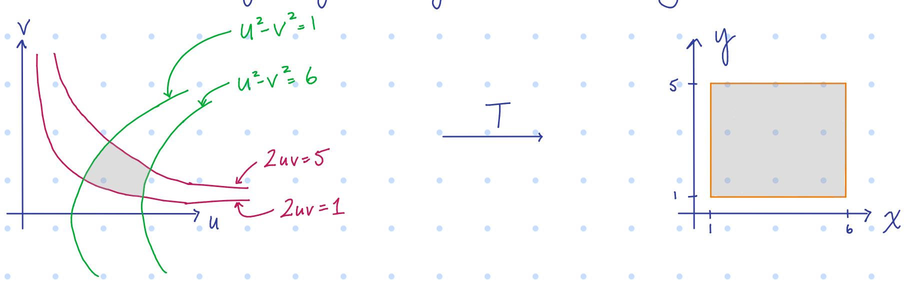
  This is a transformation that has input (u,v) and output (x,y).
  So this is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.



- T is the transformation given by $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

  Then T is a transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that turns (ρ,θ,ϕ) into (x,y,z)

- $T: (u,v) \longrightarrow (x,y)$ given by $x = u^2 - v^2$, $y = 2uv$, $1 \le x \le 6$ $1 \le y \le 5$



The Jacobian of a tranformation T given by $\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$ is the matrix determinant

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Note: If T is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, use the 2×2 matrix with z and w deleted.
$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

ex1) Calculate the Jacobian of the transformation $T: (u,v) \to (x,y)$

(a) $x = u^3 - v^2$, $y = 2uv$

(b) $x = 2e^{s-2t}$, $y = -3e^{s+2t}$

(c) $x = 3r\cos(2\theta)$, $y = 3r\sin(2\theta)$, $z = t^2$

(a) $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 3u^2 & -2v \\ 2v & 2u \end{vmatrix} = 3u^2 \cdot 2u - (-2v)\cdot 2v = \boxed{6u^3 + 4v^2}$

(b) $J = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = \begin{vmatrix} 2e^{s-2t} & -4e^{s-2t} \\ -3e^{s+2t} & -6e^{s+2t} \end{vmatrix} = -12e^{s-2t}e^{s+2t} - 12e^{s-2t}e^{s+2t} = \boxed{-24e^{2s}}$

(c) $J = \begin{vmatrix} x_r & x_\theta & x_t \\ y_r & y_\theta & y_t \\ z_r & z_\theta & z_t \end{vmatrix} = \begin{vmatrix} 3\cos(2\theta) & -6r\sin(2\theta) & 0 \\ 3\sin(2\theta) & 6r\cos(2\theta) & 0 \\ 0 & 0 & 2t \end{vmatrix}$

Note: Expanding the determinant along a row or column with a lot of zeros is very convenient

I'll use this column.

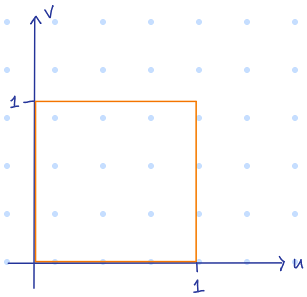
$$= 0 - 0 + 2t\left(18r\cos^2(2\theta) + 18r\sin^2(2\theta)\right)$$

$$= \boxed{36rt}$$

Let's explore what the Jacobian means.

Consider the transformation $T:(u,v)\to(x,y)$ given by $x = u+3v$, $y = u-3v$

What happens to the unit square given by $0\le u\le 1$, $0\le v\le 1$ under the tranformation T?



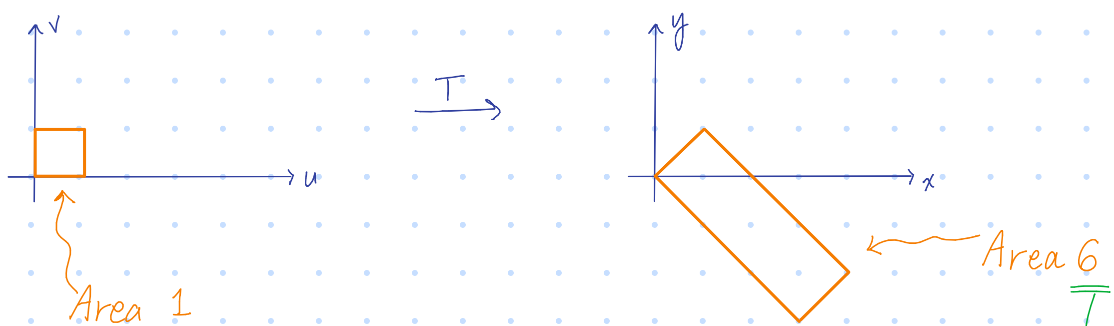
The left edge is given by $u=0$, $0\le v\le 1$
Under transformation T, we plug in $u=0$ to get $x=3v$, $y=-3v$, with $0\le v\le 1$. This is the parametric form of a line segment from $(0,0)$ to $(3,-3)$.

The right edge is given by $u=1$, $0\le v\le 1$
Under transformation T, we plug in $u=1$ to get $x=1+3v$, $y=1-3v$, with $0\le v\le 1$. This is the parametric form of a line segment from $(1,1)$ to $(4,-2)$.

The bottom edge is given by $v=0$, $0\le u\le 1$
Under transformation T, we plug in $v=0$ to get $x=u$, $y=u$, with $0\le u\le 1$. This is the parametric form of a line segment from $(0,0)$ to $(1,1)$.

The top edge is given by $v=1$, $0\le u\le 1$
Under transformation T, we plug in $v=1$ to get $x=u+3$, $y=u-3$, with $0\le u\le 1$. This is the parametric form of a line segment from $(3,-3)$ to $(4,-2)$.

Here's what T does to the unit square in the uv-plane:



The Jacobian of T is $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = \underline{\underline{-6}}$

The absolute value of the Jacobian is the scale factor for the area of a region being transformed by T.

This allows us to turn a complicated domain of integration into a simple one!

Change of variables formula for double integrals

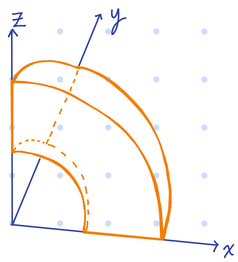
Suppose T is a transformation that maps a region S in the uv-plane to a region R in the xy-plane. Then

$$\underset{\text{complicated region} \rightarrow R}{\iint} f(x,y)\,dA = \underset{S \leftarrow \text{simple region}}{\iint} f(x(u,v),\,y(u,v)) \overbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}^{\text{absolute value of Jacobian}} du\,dv$$

Check-in 16

Evaluate $\iiint_E z\, dV$, where E lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant.

Sketch of region E



$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho\cos\phi \cdot \rho^2\sin\phi \; d\rho\, d\theta\, d\phi$$

$$\frac{\pi}{2}\int_0^{\pi/2}\int_1^2 \rho^3\cos\phi\sin\phi \; d\rho\, d\phi$$

$$\frac{\pi}{2}\int_0^{\pi/2}\cos\phi\sin\phi\left[\frac{1}{4}\rho^4\right]_1^2 d\phi$$

$$\frac{15}{4}\cdot\frac{\pi}{2}\int_0^{\pi/2}\cos\phi\sin\phi\; d\phi$$

$$\frac{15\pi}{8}\left[-\frac{1}{2}\cos^2\phi\right]_0^{\pi/2}$$

$$\boxed{15\pi/16}$$

Monday November 2

Reminders

- WebAssign 12.9
- HW 11 section 12.9 and prob A3
- Study for Check-in 17 (do examples in 12.9)

12.9 Change of Variables (cont)

Recap:

- If transformation T maps a region S in the uv-plane to the region R in the xy-plane, then $\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ is the absolute value of the Jacobian
- The Jacobian is $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$

ex 1) Suppose T is the transformation given by $x = 2u+v$, $y = u+2v$

a.) Find T^{-1}, the inverse transformation

Solve for u and v:

$$\begin{aligned} x &= 2u+v \\ -2(y &= u+2v) \\ \hline x - 2y &= -3v \\ v &= -\tfrac{1}{3}(x-2y) \end{aligned} \qquad \begin{aligned} -2(x &= 2u+v) \\ y &= u+2v \\ \hline -2x+y &= -3u \\ u &= \tfrac{1}{3}(2x-y) \end{aligned}$$

$\boxed{T^{-1} \text{ is } u = \tfrac{1}{3}(2x-y),\ v = -\tfrac{1}{3}(x-2y)}$

b) Compute the Jacobians of T and T^{-1}!

$$\text{Jacobian of } T = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

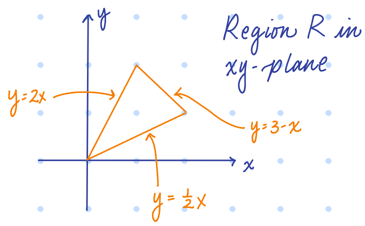
$$\text{Jacobian of } T^{-1} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

This shortcut is not in textbook

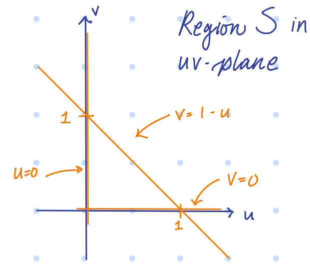
Useful fact: If the transformation T mapping a region in the uv-plane to a region in the xy-plane has Jacobian J, then the transformation T^{-1} mapping a region in the xy-plane to a region in the uv-plane has Jacobian $\frac{1}{J}$

ex2) Use the change of variables $x = 2u+v$, $y = u+2v$ to evaluate $\iint_R x-3y \, dA$ where R is the triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$.

Start with a sketch



$$T = \begin{cases} x = 2u+v \\ y = u+2v \end{cases}$$



Plug each boundary piece in to the given change of variables to find the shape of the new region S in uv-plane.

Piece 1	Piece 2	Piece 3
$y = 2x$	$y = \frac{1}{2}x$	$y = 3-x$
$u+2v = 2(2u+v)$	$u+2v = \frac{1}{2}(2u+v)$	$u+2v = 3-(2u+v)$
$u = 0$	$v = 0$	$v = 1-u$

To use the change of variables formula, we need the Jacobian of T

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

Write the new integral in terms of u,v. Use sketch to set the bounds.

$$\iint_R x-3y \, dA = \iint_S \left(2u+v-3(u+2v)\right) \cdot |3| \, dA$$

$$= 3\iint_S -u-5v \, dA$$

$$= 3\int_0^1 \int_0^{1-u} -u-5v \, dv\, du$$

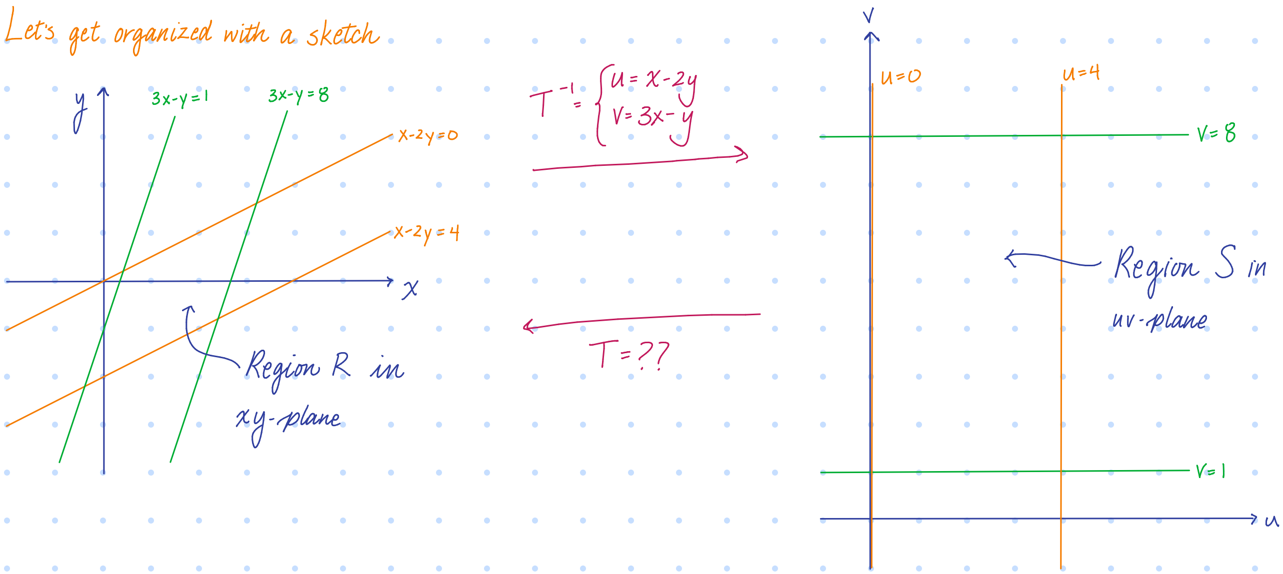
$$= \boxed{-3}$$

✳ Similar to HW11 prob A3 and popular exam question

ex 3) Use a change of variables to compute $\iint_R \frac{x-2y}{3x-y}\,dA$ where R is the region bounded by $x-2y=0$, $x-2y=4$, $3x-y=1$, $3x-y=8$

Notice $x-2y=0$ and $x-2y=4$ can be conveniently renamed $u=0$ and $u=4$ for $u=x-2y$
Similarly, $3x-y=1$ and $3x-y=8$ can be renamed $v=1$ and $v=8$ for $v=3x-y$
So let $u=x-2y$, $v=3x-y$.

Let's get organized with a sketch



We have equations for T^{-1} (from x,y to u,v) instead of T. To compute the Jacobian of T, we use the fact that the Jacobian of T is the reciprocal of the Jacobian of T^{-1}.

$$\text{Jacobian of } T^{-1} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = (1)(-1)-(3)(-2) = 5$$

$$\text{Jacobian of } T = \frac{1}{5}$$

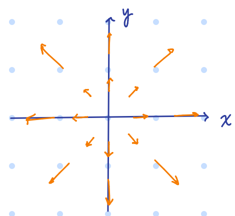
Use this to rewrite our integral

$$\iint_R \frac{x-2y}{3x-y}\,dA = \iint_S \frac{u}{v}\cdot\left|\frac{1}{5}\right|\,dA = \frac{1}{5}\int_1^8\int_0^4 \frac{u}{v}\,du\,dv = \boxed{\frac{8}{5}\ln 8}$$

13.1 Vector Fields

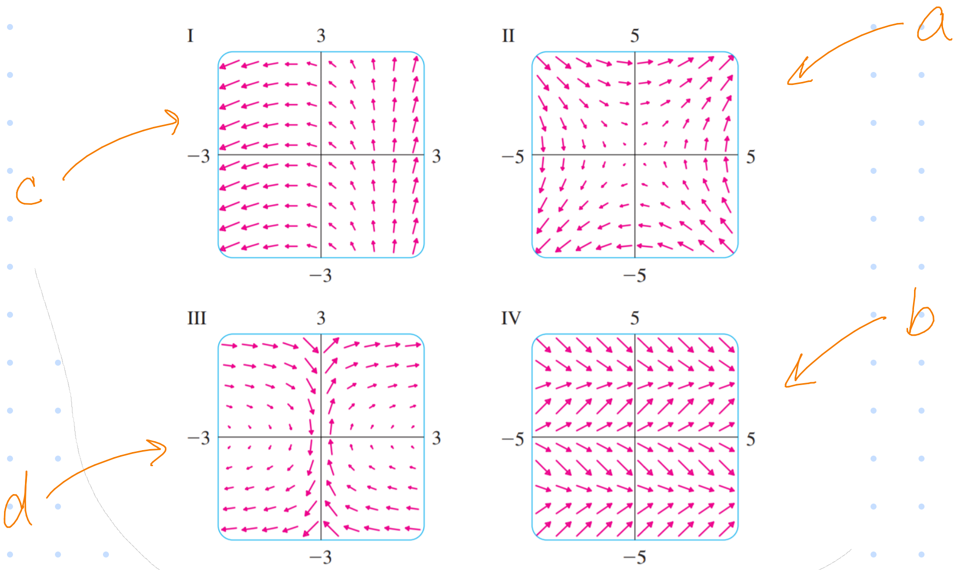
A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns a vector to each point (x,y).
A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns a vector to each point (x,y,z).

ex1) Draw the vector field $\vec{F} = \langle x, y \rangle$



"Starburst"
e-ink on tablet
C. Ng (2020)

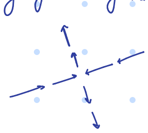
ex2) Match the vector fields (a)-(d) with the plots labeled I-IV. Give reasons for your choices.
(a) $\langle y, x\rangle$. (b) $\langle 1, \sin y\rangle$. (c) $\langle x-2, x+1\rangle$. (d) $\langle y, 1/x\rangle$.



ex3) Use VectorPlot and VectorPlot3D to find a vector field that fits each description.

a) 3D starburst
b) 2D starburst with all vectors same length
c) 2D upward flow
d) 2D swirl
e) 3D star collapse (inward starburst)

Hint for the ones below: the gradient of a surface is a vector field
f) 2D all arrows pointing away from y-axis and orthogonal to y-axis
g) 2D with one of these:



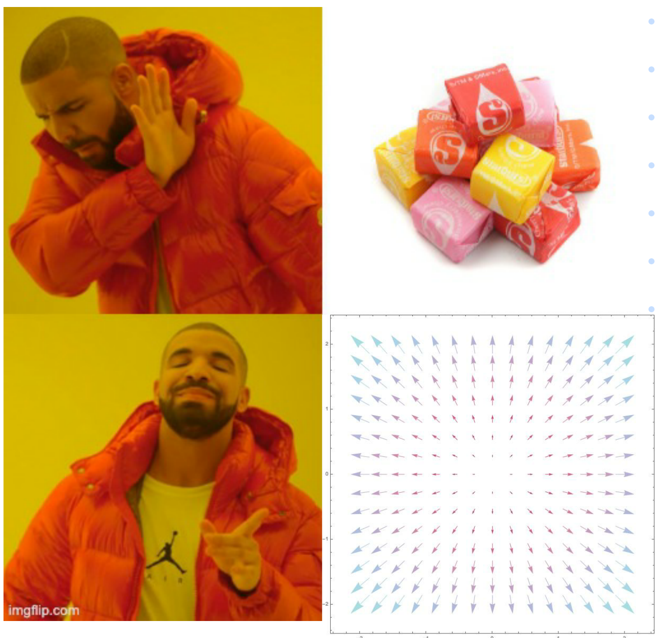
Tuesday, November 3

Reminders
- WebAssign 13.1
- HW 12 section 13.1 and problem A1 (see Piazza for technology help)
- Review: arc length formula

13.1 Vector fields

Recap

$\vec{F} = \langle x, y \rangle$ is the "starburst" vector field



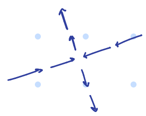
Go to student.desmos.com and use code QDC AC8

Do the "Vector Field Matching" activity

ex 3) Use VectorPlot and VectorPlot3D to find a vector field that fits each description.

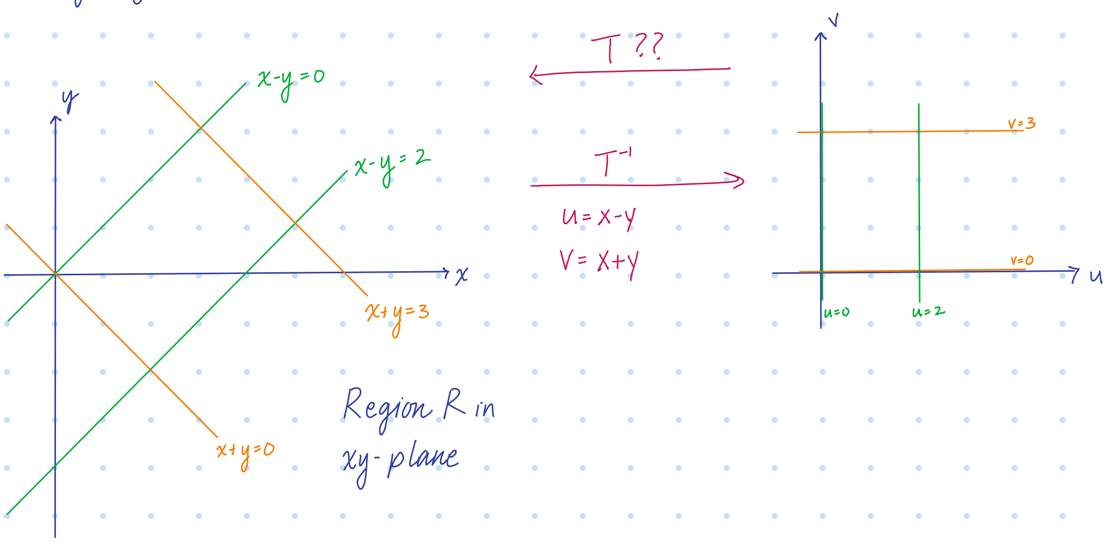
a) 3D starburst
b) 2D starburst with all vectors same length
c) 2D upward flow
d) 2D swirl
e) 3D star collapse (inward starburst)

Hint for the ones below: the gradient of a surface is a vector field
f) 2D all arrows pointing away from y-axis and orthogonal to y-axis
g) 2D with one of these:



Check-in 17

Evaluate $\iint_R (x+y)e^{x^2-y^2}\,dA$ where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, $x+y=3$ by making an appropriate change of variables



Region R in xy-plane

$$\text{Jacobian of } T^{-1} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\iint_R (x+y)e^{x^2-y^2}\,dA = \iint_S v e^{uv} \cdot \left|\tfrac{1}{2}\right| dA$$

$$= \frac{1}{2}\int_0^3 \int_0^2 v e^{uv}\,du\,dv$$

$$= \frac{1}{2}\int_0^3 v \left[\frac{1}{v}e^{uv}\right]_{u=0}^{u=2} dv$$

$$= \frac{1}{2}\int_0^3 e^{2v} - 1\,dv$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{2v} - v\right]_0^3$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{6} - 3 - \frac{1}{2} + 0\right]$$

$$= \boxed{\frac{e^6-7}{4}}$$

Wednesday, November 4

Reminders

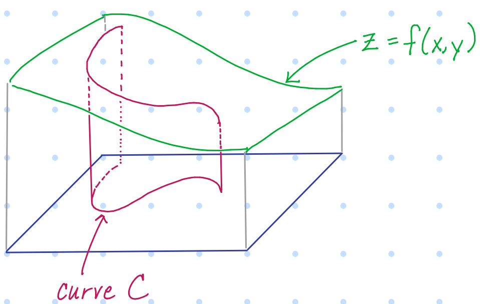
- Compile HW 11 for André
  Note: For problem A2, you do not need to write all 6 orders of integration.
  The goal is to pick the nicest one
- [Thursday] Study for Check-in 18

13.2 Line integrals

A line integral is an integral where the domain of integration is some curve C.
There are two different kinds - one where we integrate a scalar function and one where we integrate a vector function

If $z = f(x,y)$, the graph of f is a surface

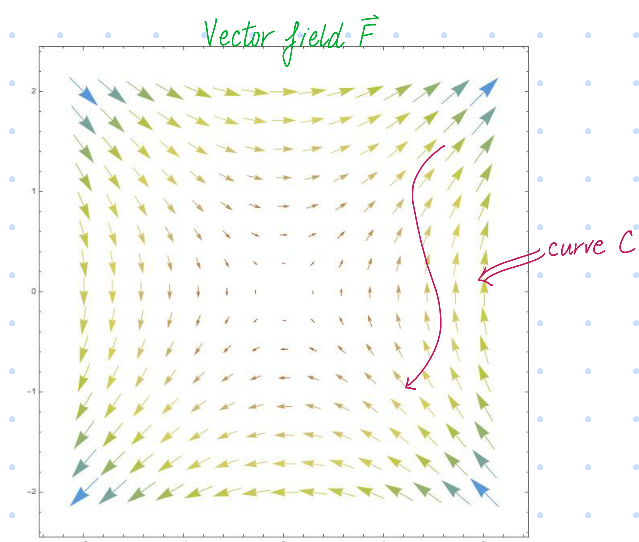
Then the line integral of $f(x,y)$ over a curve C is the area of the curved fence.



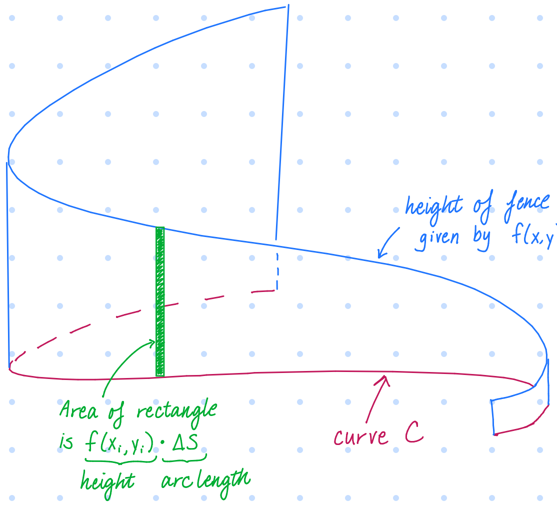
The height of the fence is $f(x,y)$.
The base of the fence is C.

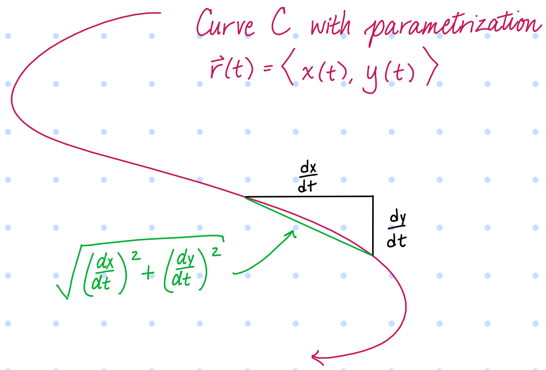
The graph of $\vec{F}(x,y)$ is a vector field

Then the line integral of $\vec{F}(x,y)$ over a curve C is the work done by $\vec{F}$ on a particle traveling on C



We discuss the line integral of a scalar function $f(x,y)$ first.



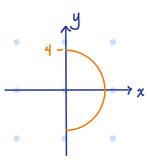


Line integral of $f(x,y)$ over C is

$$\int_C f(x,y)\,ds = \int_a^b \underbrace{f(x(t), y(t))}_{\text{(height of rectangle)}} \underbrace{\sqrt{x_t^2 + y_t^2}\,dt}_{\substack{ds \\ \text{(width of rectangle)}}}$$

ex 1) Evaluate $\int_C xy^4\,ds$ where C is the right half of $x^2+y^2=16$

Step 1: Parameterize C



$C: \vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

Step 2: Compute $ds = \sqrt{x_t^2 + y_t^2}\,dt$

$$ds = \sqrt{16\sin^2 t + 16\cos^2 t}\,dt$$
$$= 4$$

[Note: For a circle, $ds = (\text{radius})\,dt$

This can be a computational shortcut]

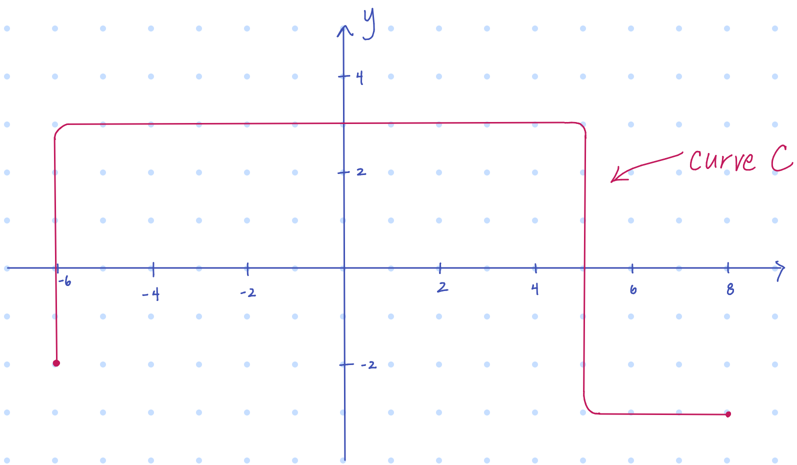
Step 3: Plug in $\vec{r}(t)$ and ds

$$\int_{-\pi/2}^{\pi/2} 4\cos t\,(4\sin t)^4 \cdot 4\,dt$$

Step 4: Compute

$$4^6 \int_{-\pi/2}^{\pi/2} \cos t \cdot \sin^4 t\,dt = 4^6 \int_{-1}^{1} u^4\,du = \boxed{\frac{4^6 \cdot 2}{5}}$$

ex 2) Suppose f is a function satisfying $7.9 \le f(x,y) \le 8.0$ for all (x,y).
Which of the numbers $0, \pm 10, \pm 50, \pm 100, \pm 200$
is closest to $\int_C f(x,y)\,ds$ for the curve C shown below?



Arc length $\approx 5 + 11 + 6 + 3$
≈ 25
Height ≈ 8
$\int_C f\,ds \approx \boxed{200}$

ex 3) Evaluate $\int_C x e^{yz}\,ds$ where C is the line segment from $(0,0,0)$ to $(1,2,3)$

Step 1: Parameterize C $\qquad C: \vec{r}(t) = \langle t, 2t, 3t \rangle \quad 0 \le t \le 1$

Step 2: Compute $ds = \sqrt{x_t^2 + y_t^2 + z_t^2}\,dt$ $\qquad ds = \sqrt{1^2 + 2^2 + 3^3}\,dt = \sqrt{14}\,dt$

Step 3: Plug in $\vec{r}(t)$ and ds $\qquad \int_0^1 t e^{6t^2} \cdot \sqrt{14}\,dt$

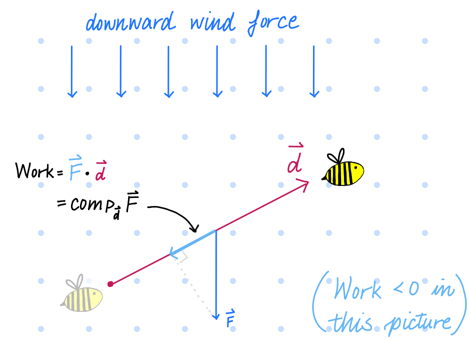
Step 4: Compute $\qquad \sqrt{14} \cdot \frac{1}{12} \int_0^6 e^u\,du = \boxed{\frac{\sqrt{14}(e^6 - 1)}{12}}$

Now let's discuss the line integral over a scalar field

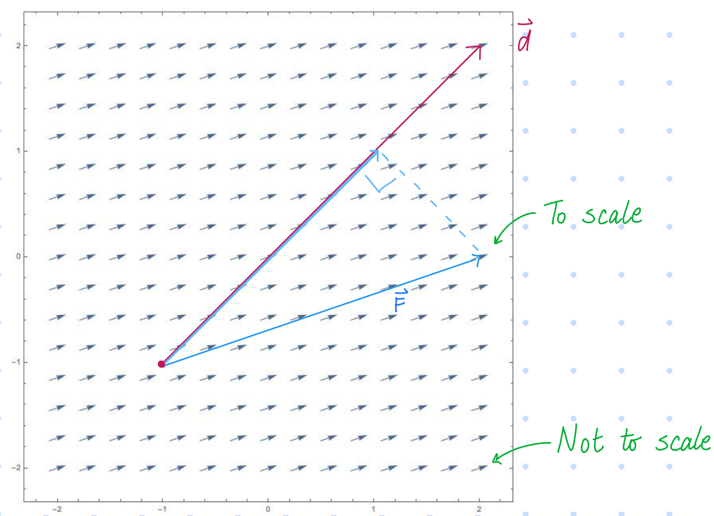
Wee bit of physics

Work = $\underbrace{\text{Force} \cdot \text{Distance}}$

These are both vectors



Here is the vector field $\langle 3,1 \rangle$.
Think of each vector as the force of some flowing wind at that point
So $\vec{F} = \langle 3,1 \rangle$ can be thought of as a constant breeze everywhere in the $\langle 3,1 \rangle$ direction.



ex 4) If a bee flies from $(-1,-1)$ to $(2,2)$ in a straight line, what is the work done on the bee by $\vec{F}$?

Length of projection $= \text{comp}_{\vec{d}} \vec{F}$

$$= \vec{F} \cdot \frac{\vec{d}}{|\vec{d}|}$$

$$= \langle 3,1 \rangle \cdot \frac{1}{3\sqrt{2}} \langle 3,3 \rangle$$

$$= 2\sqrt{2}$$

This represents the work for 1 unit length of $\vec{d}$.

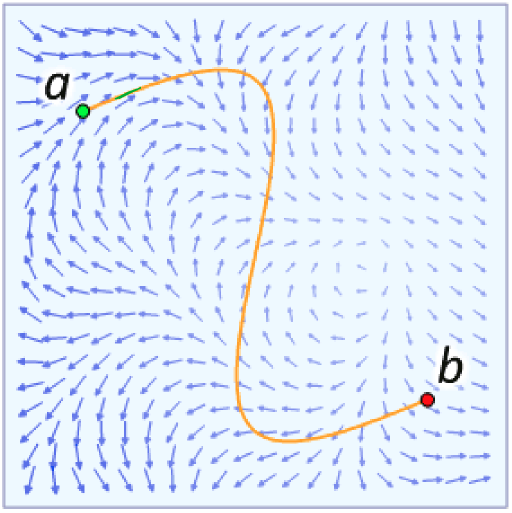
Total work $= 2\sqrt{2} \cdot$ length of $\vec{d}$

$$= 2\sqrt{2} \cdot 3\sqrt{2}$$

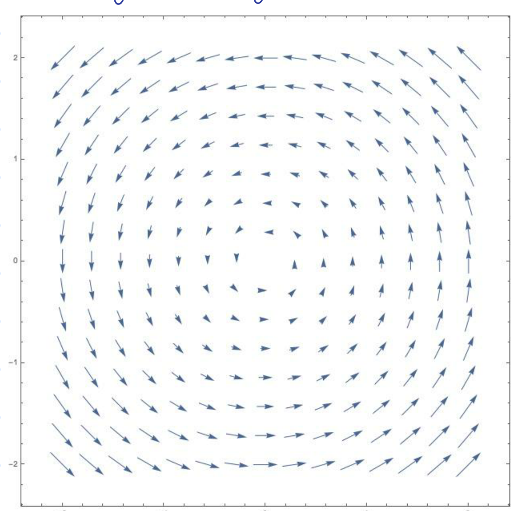
$$= 12$$

Notice that we calculated $\vec{F} \cdot \frac{\vec{d}}{|\vec{d}|}$ and then multiplied by $|\vec{d}|$, which is $\vec{F} \cdot \frac{\vec{d}}{|\vec{d}|} (|\vec{d}|) = \vec{F} \cdot \vec{d}$

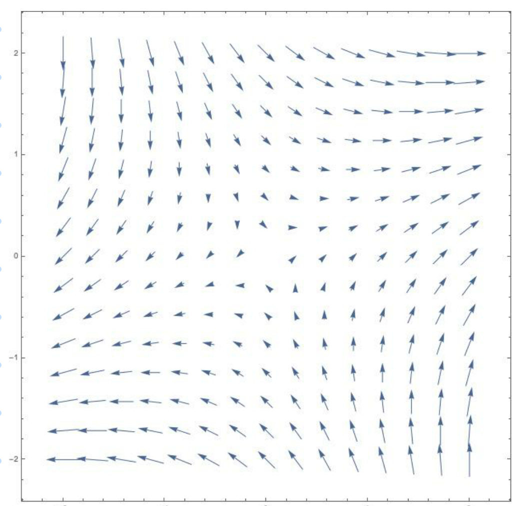
See line integrals over vector fields animation on Wikipedia



This vector field can not be the gradient of some surface



This vector field can be the gradient of some surface



Friday November 6

Reminders

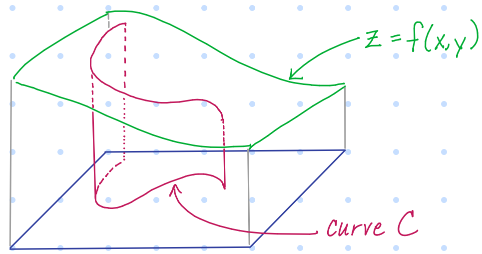
- All Week 11 WebAssign due Sunday before midnight
- HW 12 section 13.2 (esp #18, useful for Quiz 6)
- Study for Quiz 6 (sections 12.8 to 13.2)

13.2 Line integrals (cont)

Recap

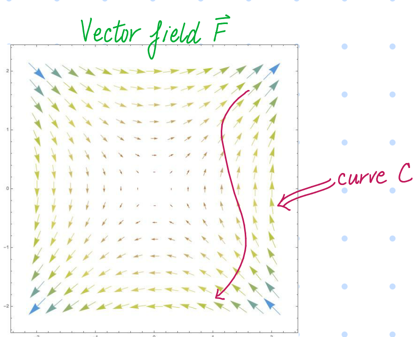
Line integral of scalar function

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|r'(t)|\,dt$$



Line integral of vector field

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$



ex 5) Let $\vec{F} = \langle 3x^2 - 6yz,\ 2y + 3xz,\ 1 - 4xyz^2 \rangle$ and evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the following paths C.

(a) The path from (0,0,0) to (1,1,1) given by $x = t$, $y = t^2$, $z = t^3$

(b) The straight line from (0,0,0) to (1,1,1)

(a) Step 1: Parametrize C

The curve C has parametrization $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$

Step 2: Compute $\vec{r}\,'(t)$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

Step 3: Compute $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$$\begin{aligned}
\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) &= \langle 3t^2 - 6t^2 \cdot t^3,\ 2t^2 + 3t \cdot t^3,\ 1 - 4t \cdot t^2 \cdot (t^3)^2 \rangle \cdot \vec{r}\,'(t) \\
&= \langle 3t^2 - 6t^5,\ 2t^2 + 3t^4,\ 1 - 4t^9 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \\
&= 3t^2 - 6t^5 + 2t(2t^2 + 3t^4) + 3t^2(1 - 4t^9) \\
&= 6t^2 + 4t^3 - 12t^{11}
\end{aligned}$$

Step 4: Integrate

$$\int_0^1 6t^2 + 4t^3 - 12t^{11}\, dt = \frac{6}{3} + \frac{4}{4} - \frac{12}{12} = \boxed{2}$$

(b) Step 1: Parametrize C

The curve C has parametrization $\vec{r}(t) = \langle t, t, t \rangle$, $0 \le t \le 1$

Step 2: Compute $\vec{r}\,'(t)$

$\vec{r}\,'(t) = \langle 1, 1, 1 \rangle$

Step 3: Compute $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$

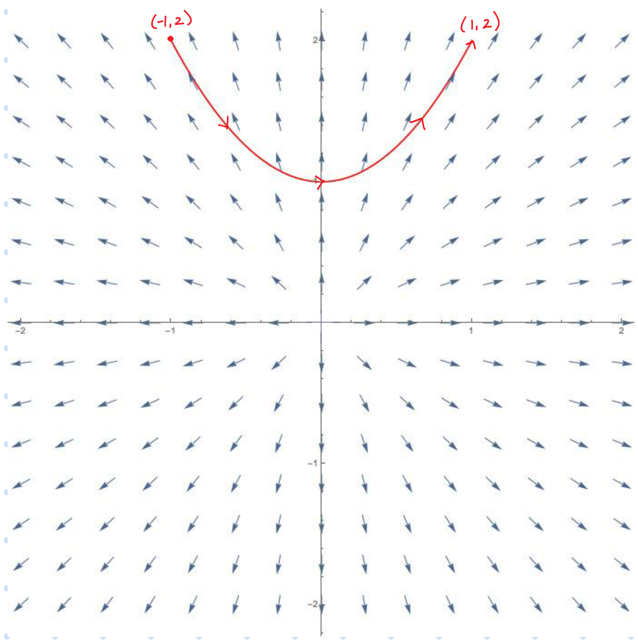
$$\begin{aligned}
\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) &= \langle 3t^2 - 6t^2,\ 2t + 3t^2,\ 1 - 4t^4 \rangle \cdot \vec{r}\,'(t) \\
&= \langle -3t^2,\ 2t + 3t^2,\ 1 - 4t^4 \rangle \cdot \langle 1, 1, 1 \rangle \\
&= 2t - 4t^4 + 1
\end{aligned}$$

Step 4: Integrate

$$\int_0^1 2t - 4t^4 + 1\, dt = \frac{2}{2} - \frac{4}{5} + 1 = \boxed{\frac{6}{5}}$$

ex 6) If $\vec{F} = \frac{x}{\sqrt{x^2+y^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j}$ and C is the parabola $y = 1+x^2$ from $(-1,2)$ to $(1,2)$,

is $\int_C \vec{F}\cdot d\vec{r}$ positive, negative, or zero?



$$\boxed{\int_C \vec{F}\cdot d\vec{r} = 0}$$

ex 7) Find the work done by the force field $\vec{F}(x,y) = x\,\vec{i} + (y+2)\,\vec{j}$ in moving an object along an arch of a cycloid $\vec{r}(t) = \langle t - \sin t,\ 1-\cos t\rangle$, $0 \le t \le 2\pi$

Step 1: Parametrize C $\qquad \vec{r}(t) = \langle t - \sin t,\ 1-\cos t\rangle,\ 0 \le t \le 2\pi$

Step 2: Compute $\vec{r}\,'(t)$ $\qquad \vec{r}\,'(t) = \langle 1-\cos t,\ \sin t\rangle$

Step 3: Compute $\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)$

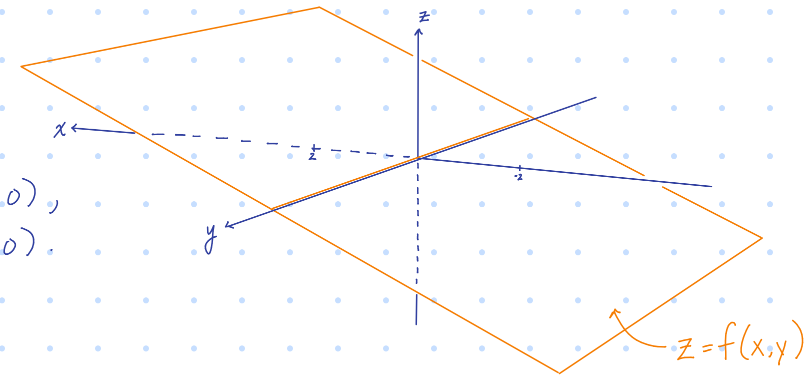
$$\begin{aligned}
\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) &= \langle t-\sin t,\ 1-\cos t+2\rangle \cdot \vec{r}\,'(t) \\
&= \langle t-\sin t,\ 3-\cos t\rangle \cdot \langle 1-\cos t,\ \sin t\rangle \\
&= t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t \\
&= t - t\cos t + 2\sin t
\end{aligned}$$

Step 4: Integrate

$$\int_0^{2\pi} t - t\cos t + 2\sin t\ dt = \left[\tfrac{1}{2}t^2\right]_0^{2\pi} - \left(\left[t\sin t\right]_0^{2\pi} - \int_0^{2\pi}\sin t\ dt\right) + 0$$

$$= \boxed{2\pi^2}$$

Check-in 18



Suppose $f(x,y) = x$, $\vec{F} = \nabla f$,

C_1 is the line from $(-2,0)$ to $(2,0)$,

C_2 is the line from $(0,0)$ to $(-4,0)$.

(a) Is $\int_{C_1} f\,ds$ positive, negative, or zero? 0

(b) Is $\int_{C_2} f\,ds$ positive, negative, or zero? $-$

(c) Is $\int_{C_1} \vec{F}\cdot d\vec{r}$ positive, negative, or zero? $+$

(d) Is $\int_{C_2} \vec{F}\cdot d\vec{r}$ positive, negative, or zero? $-$

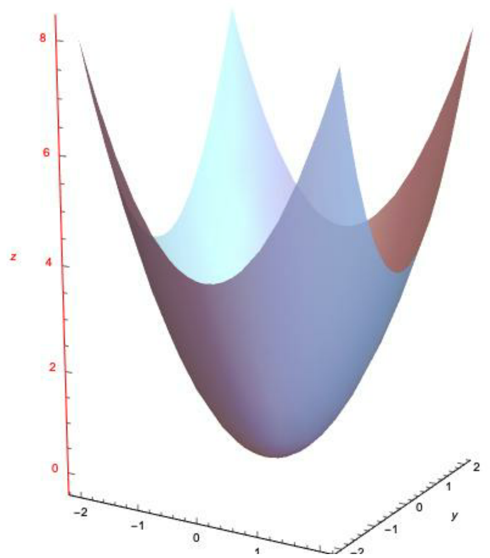
Monday November 9

Reminders
- Do Check-in 19 on Canvas before midnight tonight
- Study for Quiz 6

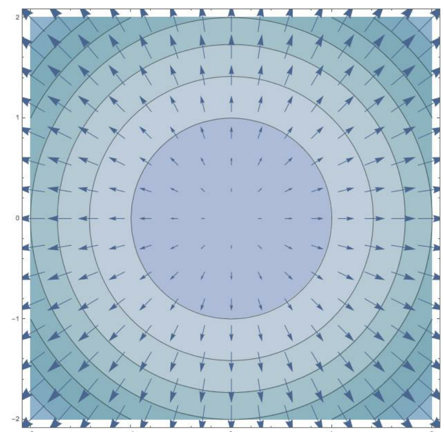
13.3 Fundamental Theorem of Line Integrals

Recap

Let $z=f(x,y)$ be a scalar-valued function. Then $\nabla f = \langle f_x, f_y \rangle$ is a vector field in $\mathbb{R}^2$ whose vectors point in the direction of steepest increase



Graph of $f(x,y)=x^2+y^2$

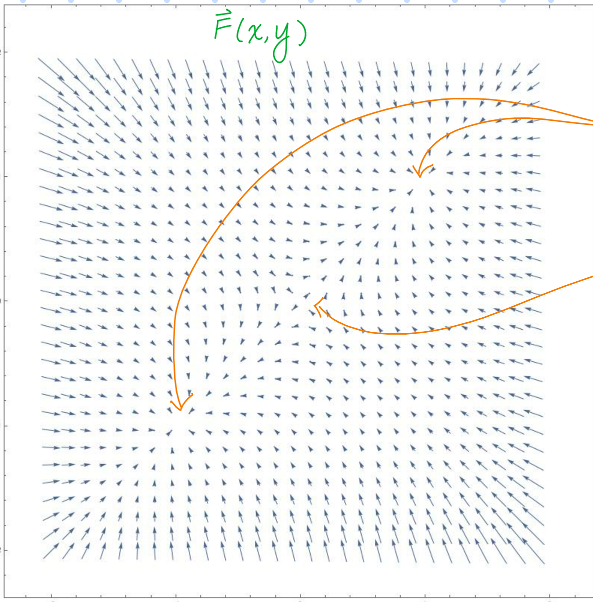


Level sets $x^2+y^2=k$ form contour plot
Gradient $\nabla f = \langle 2x, 2y \rangle$ is the vector field.

If a vector field $\vec{F}$ is the gradient vector field of a scalar function f, then we say $\vec{F}$ is a conservative vector field. The scalar function f is the potential function of $\vec{F}$.

Let's look at why we use these terms.

ex 1) The vector field $\vec{F}$ is graphed below. By inspection, does $\vec{F}$ appear to be conservative?



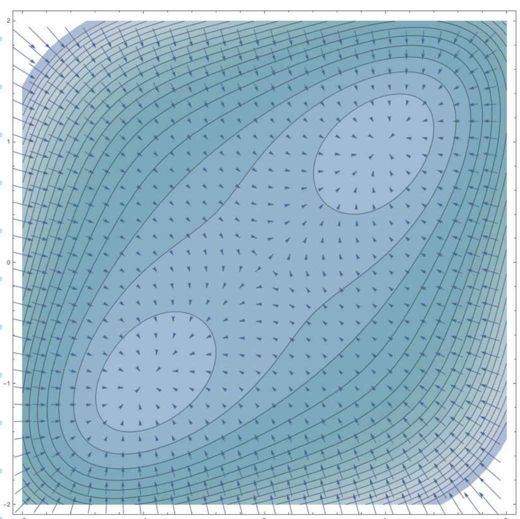
Just eyeballing this graph, if it were the gradient of some surface, there's local maxima here

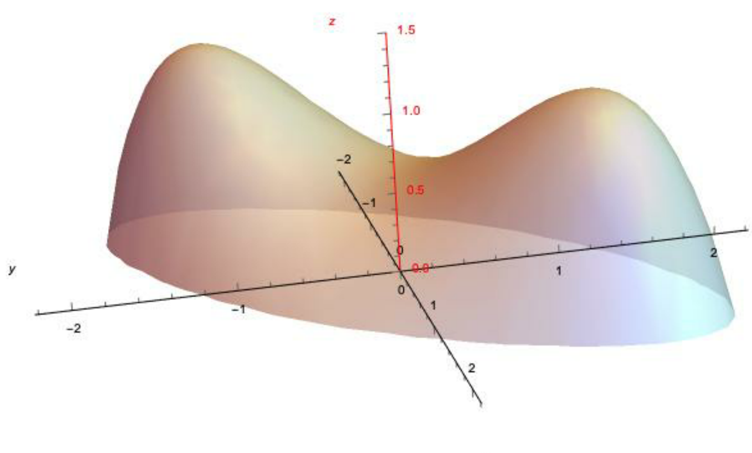
sadde point here

uphill toward the two local maxima everywhere else

Yes, this looks like it could be the gradient of a two-hump surface.

Here it is:





These are the contour plot and gradient field of the surface $z = f(x,y)$ on the right

This vector field is conservative in the sense that energy is conserved

This is the graph of the potential function of the vector field on the left.

This is called a potential function because it represents the potential energy, or electrostatic potential, or some other kind of potential

If $\vec{F}$ is a conservative vector field, we can actually find its potential function f.
This is analogous to finding the antiderivative in Calc 1.

ex 2) Let $\vec{F}(x,y) = (2x-3y)\,\vec{i} + (-3x+4y-8)\,\vec{j}$ and find $f(x,y)$ such that $\vec{F} = \nabla f$.

We are looking for $f(x,y)$ such that its gradient is $\vec{F}$.
So we want

$$\nabla f = \vec{F}$$
$$\langle f_x, f_y \rangle = \langle 2x-3y, -3x+4y-8 \rangle$$

$f_x = 2x - 3y$
integrate with respect to x
$f = x^2 - 3yx + \underbrace{c(y)}_{\text{some unknown terms that contain only } y\text{'s and no } x\text{'s}}$

$f_y = -3x + 4y - 8$
integrate with respect to y
$f = -3xy + 2y^2 - 8y + \underbrace{d(x)}_{\text{some unknown terms that contain only } x\text{'s and no } y\text{'s}}$

Compare the two versions of f to figure out what f should be.
Be sure to include every term and ignore duplicates.

$f = x^2 - 3yx + c(y)$ $\qquad$ $f = -3xy + 2y^2 - 8y + d(x)$

$$f(x,y) = x^2 - 3xy + 2y^2 - 8y + \underbrace{C}_{\text{This can be any constant}}$$

The reasons that we care about conservative vector fields are
(1) many physical phenomena have conservative force fields (gravity, electrostatics, etc)
(2) it makes line integrals way easier!

Fundamental Theorem of Line Integrals

Let C be a smooth curve with parametrization $\vec{r}(t)$ for $a \le t \le b$.
Let f be a differentiable function whose gradient ∇f is continuous on C.
Then

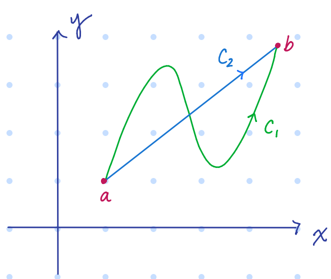
$$\int_C \nabla f \cdot d\vec{r} = f(\underbrace{\vec{r}(b)}_{\text{plug in end pt}}) - f(\underbrace{\vec{r}(a)}_{\text{plug in start pt}})$$

☆ Useful fact based on the Fundamental Theorem of Line Integrals: ☆

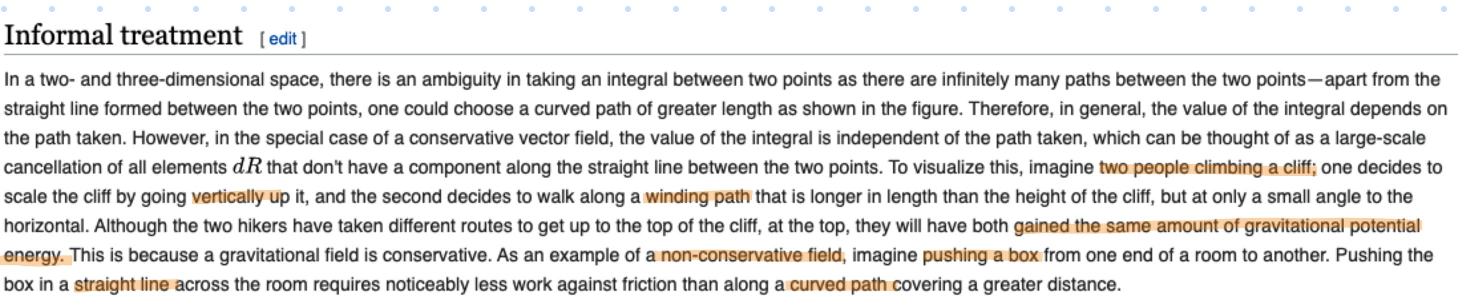
When $\vec{F}$ is conservative, the actual path of integration doesn't matter at all!
The only thing that matters is the value of f at the endpoints!
This is called path independence.

If $\vec{F} = \nabla f$, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$



Wikipedia article on conservative vector field gives intuition about path independence.



Informal treatment [edit]

In a two- and three-dimensional space, there is an ambiguity in taking an integral between two points as there are infinitely many paths between the two points—apart from the straight line formed between the two points, one could choose a curved path of greater length as shown in the figure. Therefore, in general, the value of the integral depends on the path taken. However, in the special case of a conservative vector field, the value of the integral is independent of the path taken, which can be thought of as a large-scale cancellation of all elements dR that don't have a component along the straight line between the two points. To visualize this, imagine two people climbing a cliff; one decides to scale the cliff by going vertically up it, and the second decides to walk along a winding path that is longer in length than the height of the cliff, but at only a small angle to the horizontal. Although the two hikers have taken different routes to get up to the top of the cliff, at the top, they will have both gained the same amount of gravitational potential energy. This is because a gravitational field is conservative. As an example of a non-conservative field, imagine pushing a box from one end of a room to another. Pushing the box in a straight line across the room requires noticeably less work against friction than along a curved path covering a greater distance.

ex 3) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}$ is a conservative vector field given by $\vec{F} = x^2\vec{i} + y^2\vec{j}$ and C is the arc of the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$

$\vec{F}$ is conservative, so $\vec{F}$ is the gradient of some function f. If we find f, then we can use the Fundamental Theorem of Line Integrals

1) Find f

$f_x = x^2$ $\qquad$ $f_y = y^2$

$f = \frac{1}{3}x^3 + c(y)$ $\qquad$ $f = \frac{1}{3}y^3 + d(x)$

$$f(x,y) = \frac{1}{3}(x^3 + y^3) + C$$

2) Use FTLI

$$\int_C \vec{F}\cdot dr = \int_C \nabla f \cdot d\vec{r}$$

$$= f(\text{end pt}) - f(\text{start pt})$$

$$= f(2,8) - f(-1,2)$$

$$= \frac{1}{3}(8^3 + 2^3) - \frac{1}{3}(2^3 + (-1)^3)$$

$$= \frac{8^3}{3} + \frac{1}{3}$$

$$= \frac{513}{3} = \boxed{171}$$

Some vocabulary: A curve is simple if it does not intersect itself
A curve is closed if it starts and stops at the same point

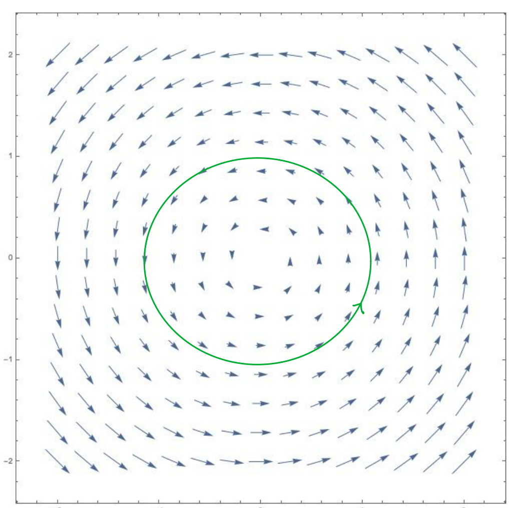
Which of these curves are closed? simple?

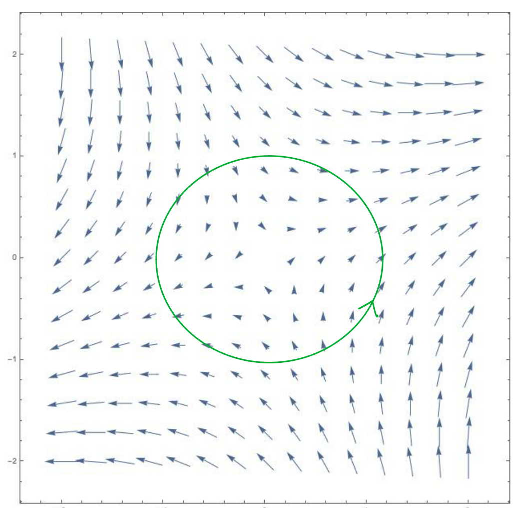
simple not closed | not simple closed | not simple not closed | simple closed

Useful fact based on previous useful fact:

☆ If $\vec{F}$ is conservative and C is a simple closed curve, then $\int_C \vec{F}\cdot d\vec{r} = 0$ ☆

Which of these vector fields is not conservative?





This vector field not conservative because $\int_C \vec{F}\cdot d\vec{r}$ is positive and not zero when C is a counter-clockwise loop around the origin

This vector field looks like it can be conservative, but hard to be sure based on graph alone.

ex 4) (a) If $\vec{F}(x,y,z) = e^y\,\vec{i} + xe^y\,\vec{j} + (z+1)e^z\,\vec{k}$ and $\vec{F}$ is conservative, find its potential function f.

$f_x = e^y$

$f = xe^y + \underset{\sim\sim\sim}{c(y,z)}$

unknown terms with y, z but no x.

$f_y = xe^y$

$f = xe^y + d(x,z)$

$f_z = (z+1)e^z$

$f = (z+1)e^z - \int e^z\,dz$

$= ze^z + h(x,y)$

$$\boxed{f(x,y,z) = xe^y + ze^z + C}$$

(b) If C_1 is given by $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ for $0 \le t \le 1$, evaluate $\int_{C_1} \vec{F}\cdot d\vec{r}$

$\vec{F}$ is conservative, so use FTLI

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} \nabla f \cdot d\vec{r}$$

$$= f(1,1,1) - f(0,0,0)$$

$$= 2e - 0$$

$$= \boxed{2e}$$

(c) If C_2 is given by $\vec{r}_2(t) = \langle \sin\frac{\pi t}{2}, t, te^{t^2} \rangle$ for $0 \le t \le 1$, evaluate $\int_{C_2} \vec{F}\cdot d\vec{r}$

Notice $\vec{r}_2(0) = (0,0,0)$ and $\vec{r}_2(1) = (1,1,1)$. Since $\vec{F}$ is conservative, $\int_C \vec{F}\cdot d\vec{r}$ is path independent

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} = \boxed{2e}$$

(d) If C_3 is given by $\vec{r}_3(t) = \langle \sin t, \cos t, 0 \rangle$ for $0 \le t \le 2\pi$, evaluate $\int_{C_3} \vec{F}\cdot d\vec{r}$

Since $\vec{F}$ is conservative and C_3 is a simple closed loop, $\int_{C_3} \vec{F}\cdot d\vec{r} = \boxed{0}$

Tuesday, November 10

Reminders

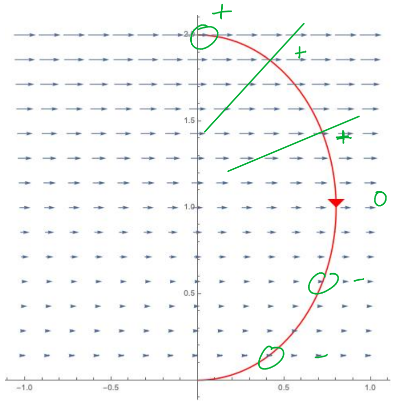
- [Important] Do Quiz 6 on Canvas between 7 and 10 pm
  - A timer is strongly recommended
  - Maybe do the free response question first?

Name: ____

1. Determine whether each statement is TRUE or FALSE
   (a) The integral $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ gives the volume of 1/4 of a sphere.
   (b) The integral $\int_0^{2\pi}\int_0^2\int_r^2 r\, dz\, dr\, d\theta$ represents the volume enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$
   (c) If the work done by a force $\vec{\mathbf{F}}$ on an object moving along a curve is W, then for an object moving along the curve in the opposite direction, the work done by $\vec{\mathbf{F}}$ will be $-W$.
   (d) If a particle moves along a curve C, the total work done by a force $\vec{\mathbf{F}}$ on the object is independent of how quickly the particle moves.
   (e) If f is a scalar-valued function, then $\int_{-C} f\, ds = -\int_C f\, ds$
   (f) The line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is a vector.
2. Evaluate $\iiint_E x\, dV$, where E is enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.
3. Find the volume of the part of the ball $\rho \le 5$ that lies between the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.
4. Evaluate $\iint_R (x^2 - xy + y^2)\, dA$ where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$. Use the transformation given by $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$
5. Use an appropriate change of variables to evaluate $\iint_R \frac{x-y}{x+y}\, dA$ where R is the square with vertices $(0, 2)$, $(1, 1)$, $(2, 2)$, and $(1, 3)$.
6. Sketch the following vector fields in the xy-plane.
   (a) $\vec{\mathbf{F}}(x, y) = \langle y, 0\rangle$
   (b) $\vec{\mathbf{F}}(x, y) = \langle 2, 3\rangle$
   (c) $\vec{\mathbf{F}}(x, y) = \langle -y, x\rangle$
7. Calculate the following line integrals.
   (a) $\int_C 3x^2 - 2y\, ds$ where C is the segment from $(3, 6)$ to $(1, -1)$
   (b) $\int_C 2x^3\, ds$ where C is the portion of $y = x^3$ from $x = 2$ to $x = -1$.
   (c) $\int_C 2yx^2 - 4x\, ds$ where C is the lower half of the circle centered at the origin of radius 3.
   (d) $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}} = \langle y^2, 3x - 6y\rangle$ and C is the line segment from $(3, 7)$ to $(0, 12)$.
   (e) $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}} = \langle x + y, 1 - x\rangle$ and C is the portion of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the fourth quadrant with counterclockwise orientation.

8. For each vector field $\vec{\mathbf{F}}$ and curve C shown below, is the value of $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ positive, negative, or zero?

(a)

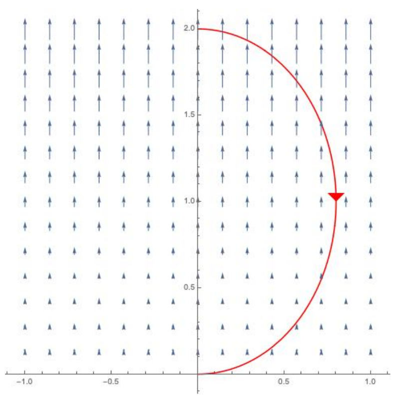


+

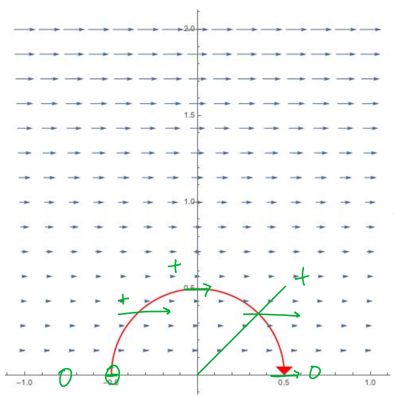
−

$\boxed{\text{pos}}$

(b)

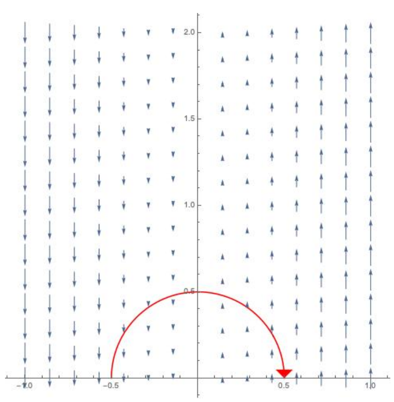


(c)



$\boxed{\text{pos}}$

(d)



Answers
(1) F T T T F F
(2) $65\pi/4$
(3) $\dfrac{\sqrt{3}-1}{3}125\pi$
(4) $4\pi/\sqrt{3}$
(5) $-\ln 2$
(6) Use Mathematica function VectorPlot[] or this online plotter [`https://academo.org/demos/vector-field-plotter/`] to check your answers.
(7) (a) $8\sqrt{53}$ *(b)* $(145^{3/2}-10^{3/2})/27 \approx 63.4966$ *(c)* -108 *(d)* $-1079/2$ *(e)* $5-3\pi$
(8) pos, neg, pos, neg

4. Evaluate $\iint_R (x^2 - xy + y^2)\, dA$ where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$. Use the transformation given by $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$

Formula: $\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

Compute Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{vmatrix} = \frac{4}{\sqrt{3}}$

Convert integrand to u,v: $x = \sqrt{2}u - \sqrt{2/3}\,v$, $y = \sqrt{2}u + \sqrt{2/3}\,v$

$$x^2 - xy + y^2 = \left(\sqrt{2}u - \sqrt{2/3}\,v\right)^2 - 2\left(\sqrt{2}u - \sqrt{2/3}\,v\right)\left(\sqrt{2}u + \sqrt{2/3}\,v\right) + \left(\sqrt{2}u + \sqrt{2/3}\,v\right)^2$$

$$= 2u^2 + 2v^2$$

Convert region R to u,v: $x^2 - 2xy - y^2 = 2$

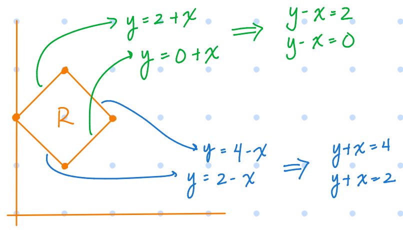
$2u^2 + 2v^2 = 2$

$u^2 + v^2 = 1$ ← new region is inside unit circle.

Write new integral: $\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} (2u^2 + 2v^2)\left|\frac{4}{\sqrt{3}}\right| du\,dv$

(you may convert to polar to do final computations)

5. Use an appropriate change of variables to evaluate $\iint_R \frac{x-y}{x+y}\, dA$ where R is the square with vertices $(0,2)$, $(1,1)$, $(2,2)$, and $(1,3)$.



Let $u = y - x$, $v = y + x$ be our new region S.

$0 \le u \le 2 \qquad 2 \le v \le 4$

Formula: $\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$

Compute Jacobian: $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ hard to compute because our transformation is not in the form $x = \_\_$, $y = \_\_$

We'll compute $\left| \frac{\partial(u,v)}{\partial(x,y)} \right|$ instead

$$\left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = -\frac{1}{2}$$

Convert integrand: $\frac{x-y}{x+y} = \frac{-u}{v}$

Write new integral: $\int_2^4 \int_0^2 -\frac{u}{v} \left| -\frac{1}{2} \right| du\, dv = -\frac{1}{2} \int_2^4 \int_0^2 \frac{u}{v}\, du\, dv$

Do integral: ... ☺

(b) $\int_C 2x^3\, ds$ where C is the portion of $y = x^3$ from $x = 2$ to $x = -1$.

Formula $\int_C f\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$

Parametrize C: $\vec{r}(t) = \langle t, t^3 \rangle \quad -1 \le t \le 2$

Compute $|\vec{r}\,'(t)|$: $\vec{r}\,'(t) = \langle 1, 3t^2 \rangle$

$|\vec{r}\,'(t)| = \sqrt{1 + 9t^4}$

Plug $\vec{r}(t)$ into f: $f(\vec{r}(t)) = 2t^3$

Write integral: $\int_{-1}^{2} 2t^3 \sqrt{1+9t^4}\, dt$

Quick curve parametrization review

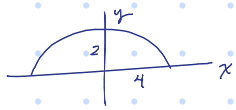
bottom half of circle of radius 1

$\vec{r}(t) = \langle \cos t, \sin t \rangle \quad \pi \le t \le 2\pi$

left half of circle of radius 5

$\vec{r}(t) = \langle 5\cos t, 5\sin t \rangle \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

top half of this ellipse



$\vec{r}(t) = \langle 4\cos t, 2\sin t \rangle \quad 0 \le t \le \pi$

line segment from $(1,2)$ to $(6,13)$

$\vec{r}(t) = \langle 1+5t, 2+11t \rangle \quad 0 \le t \le 1$

line segment from $(0,6)$ to $(5,-2)$

$\vec{r}(t) = \langle 5t, 6-8t \rangle \quad 0 \le t \le 1$

line segment from $(10,7)$ to $(4,6)$

$\vec{r}(t) = \langle 10-6t, 7-t \rangle \quad 0 \le t \le 1$

$y = \sqrt{x-4}$ from $(4,0)$ to $(20,4)$

$\vec{r}(t) = \langle t, \sqrt{t-4} \rangle \quad 4 \le t \le 20$

$x = y^3$ from $(-1,-1)$ to $(8,2)$

$\vec{r}(t) = \langle t^3, t \rangle \quad -1 \le t \le 2$

$x = e^y$ from $x = e$ to $x = e^3$

$r(t) = \langle e^t, t \rangle \quad 1 \le t \le 3$

$y = 4\sin(x^2)$ from $x = 0$ to $x = 2\pi$

$\vec{r}(t) = \langle t, 4\sin(t^2) \rangle \quad 0 \le t \le 2\pi$

How to pick appropriate transformation

Textbook section 12.9 exercises

23) $\iint_R \frac{x-2y}{3x-y}\, dA$ R parallelogram enclosed by

$x-2y=0$
$x-2y=4$ } These look similar to each other
$3x-y=1$
$3x-y=8$

$u = x-2y \qquad 0 \le u \le 4$
$v = 3x-y \qquad 1 \le v \le 8$

new integrand is $\frac{u}{v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

24) $\iint_R (x+y)\, e^{x^2-y^2}\, dA$ R rectangle enclosed by

$[\, x-y=0 \qquad x-y=2\,]$ similar
$[\, x+y=0 \qquad x+y=3\,]$ similar

$u = x-y \qquad 0 \le u \le 2$
$v = x+y \qquad 0 \le v \le 3$

new integrand is $v\, e^{uv}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

26) $\iint_R \sin(9x^2+4y^2)\, dA$ R region in first quadrant bounded ellipse $9x^2+4y^2=1$

Circles are easier than ellipses, so to turn $9x^2+4y^2=1$ into $u^2+v^2=1$ we pick $u=3x$, $v=2y$

new integrand is $\sin(u^2+v^2)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

Wednesday November 11

Reminders

- Submit Quiz 6 corrections on Canvas by 10 pm
- Compile HW 12 for André
  - For problem A1, see Tasks post on Piazza for Mathematica help

13.3 Fundamental Theorem of Line Integrals (cont)

Recap

If $\vec{F}$ is a conservative vector field...
- $\vec{F} = \nabla f$ for some scalar function f
- $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$
- $\int_C \vec{F} \cdot d\vec{r}$ is path independent
- and C is a simple closed curve, then $\int_C \vec{F} \cdot d\vec{r} = 0$

These properties are so nice, but how can we tell that $\vec{F}$ is conservative without being told? To answer this, we need some new vocabulary.

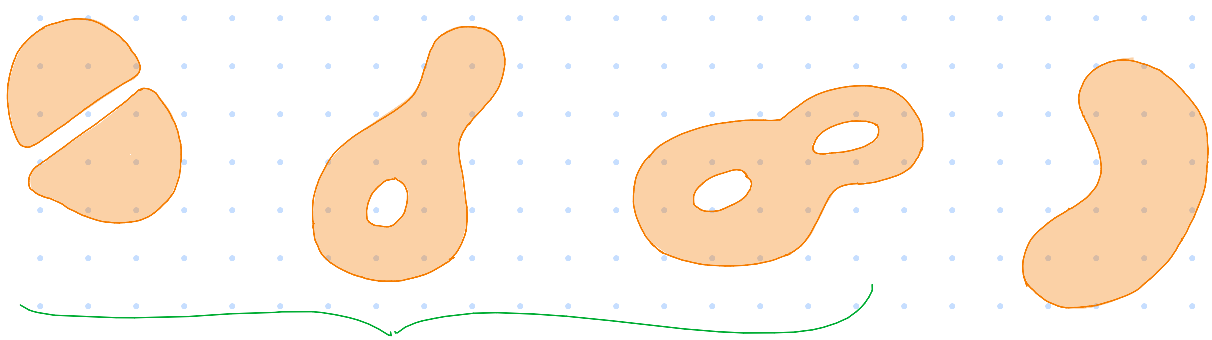
- A region D in $\mathbb{R}^2$ is <u>open</u> if it does not include the boundary

  ex) Which region is open?

$x^2+y^2 \le 1$ $\qquad$ $x^2+y^2 < 1$

not open $\qquad$ open

- A region D is <u>simply-connected</u> if it consists of one piece with no holes



not simply connected $\qquad$ simply connected

Testing for conservativeness

Let $\vec{F} = \langle P, Q \rangle$ be a vector field on an open, simply-connected region D. Suppose P and Q have continuous first derivatives. Then

if $P_y = Q_x$ everywhere on D, then $\vec{F}$ is conservative

ex 5) Is $\vec{F} = e^x \cos y\, \vec{i} + e^x \sin y\, \vec{j}$ a conservative vector field?

$P = e^x \cos y$ $\qquad Q = e^x \sin y$

$P_y = -e^x \sin y$ $\qquad Q_x = e^x \sin y$

$P_y \neq Q_x$ so $\vec{F}$ is not conservative

ex 6) Is $\vec{F} = (2xy + y^{-2})\, \vec{i} + (x^2 - 2xy^{-3})\, \vec{j}$ a conservative vector field when $y > 0$?

$P = 2xy + y^{-2}$ $\qquad Q = x^2 - 2xy^{-3}$

$P_y = 2x - 2y^{-3}$ $\qquad Q_x = 2x - 2y^{-3}$

$P_y = Q_x$ for all x, y except $y = 0$. But our domain is $y > 0$, so we don't need to worry about $y = 0$. Lastly, the domain $y > 0$ is open and simply-connected. So we conclude $\vec{F}$ is conservative.

This problem demonstrates that we must check that $P_y = Q_x$ <u>and</u> D is open and simply-connected before concluding that $\vec{F}$ is conservative.

ex 7) (a) Let $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$. Does $P_y = Q_x$?

$P = \frac{-y}{x^2+y^2}$ $\qquad$ $Q = \frac{x}{x^2+y^2}$

$P_y = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$ $\qquad$ $Q_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$ $\qquad$ $\boxed{\text{Yes, except at } (0,0)}$

$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$ $\qquad$ $= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

$= \frac{y^2-x^2}{(x^2+y^2)^2}$ $\qquad$ $= \frac{y^2-x^2}{(x^2+y^2)^2}$

(b) Compute $\int_C \vec{F} \cdot d\vec{r}$ where C is the unit circle oriented counterclockwise.
Is $\vec{F}$ conservative?

Parametrize C $\qquad$ $\vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le 2\pi$

Compute $\vec{r}\,'(t)$ $\qquad$ $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

Compute $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ $\qquad$ $\langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle$

$\sin^2 t + \cos^2 t$

1

Write integral $\qquad$ $\int_0^{2\pi} 1 \, dt = \boxed{2\pi}$

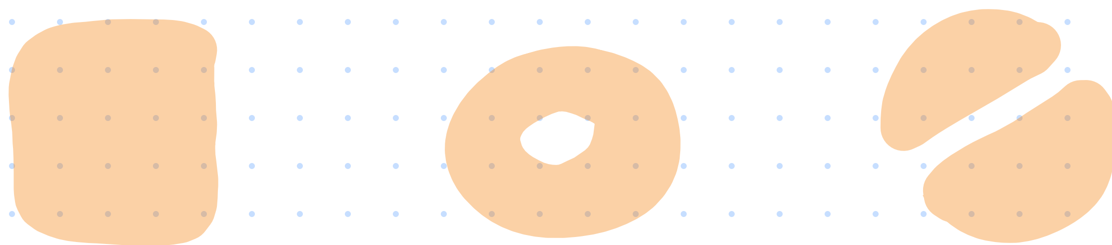
$\vec{F}$ can't be conservative because if $\vec{F}$ were conservative, the integral $\int_C \vec{F} \cdot d\vec{r}$ for a simple closed curve C should be zero.

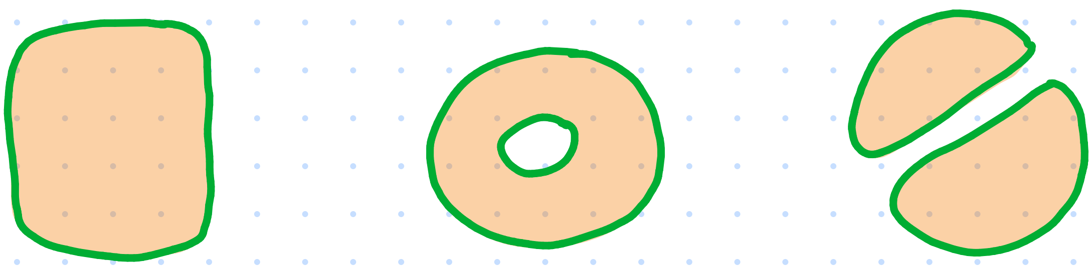
13.4 Green's Theorem

First, some new vocabulary!

- Let D be a region in $\mathbb{R}^2$. Then the boundary of D, denoted ∂D, is the curve that encloses D

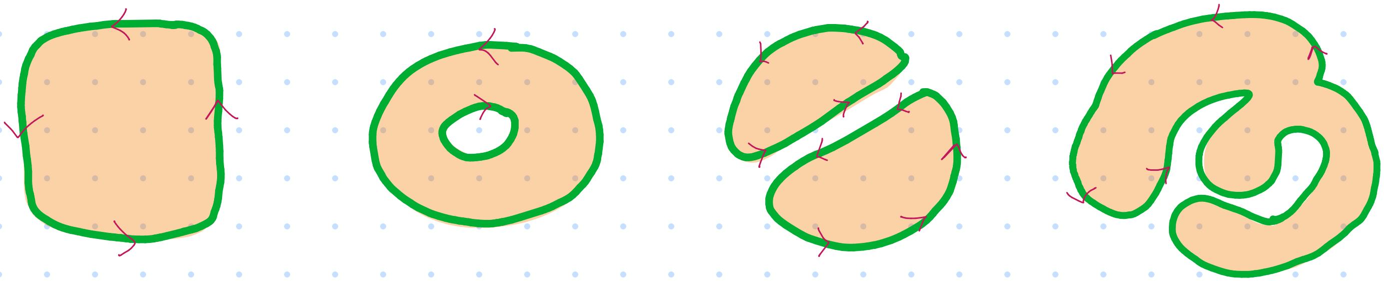
  ex) Trace ∂D for each region D





- Let C be a simple closed curve in $\mathbb{R}^2$ and let D be the region enclosed by C. We say C is positively-oriented if D is on the left of C as we travel along C.

  ex) Mark the direction of C that makes it positively-oriented



Green's Theorem

Let C be a positively-oriented, piecewise smooth, simple closed curve.
Let D be the region enclosed by C.
Let $\vec{F} = \langle P, Q \rangle$ and P, Q have continuous partial derivatives on an open region containing D.
Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$$

Alternative notations: (1) $\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$

$\oint$ means "line integral over closed curve"

(2) $\oint_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$

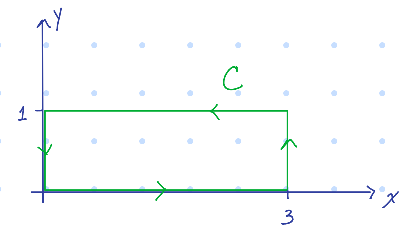
tiny arrow gives orientation

My favorite

(3) $\int_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$

Least favorite

The magic of Green's Theorem is that it gives us the power to switch between a line integral and a double integral. We can choose whatever is most convenient ex 1) Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle xy, x^2 \rangle$ over the rectangular curve C.



Hmm, if $\vec{F}$ is conservative, then this integral is zero! But $P_y = x$ and $Q_x = 2x$ so $\vec{F}$ isn't conservative. Shucks.

The curve C is complicated (made of 4 separate lines!) but the region inside is nice. Use Green's theorem!

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$$

$$= \iint_D 2x - x \, dA$$

$$= \iint_D x \, dA$$

$$= \int_0^3 \int_0^1 x \, dy\, dx$$

$$= \frac{1}{2} x^2 \Big|_0^3$$

$$= \boxed{9/2}$$

ex 2) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = \langle e^x + x^2 y, e^y - xy^2 \rangle$ and C is the circle of radius 5 oriented clockwise.

Hmm, if $\vec{F}$ is conservative, then this integral is zero! But $P_y = x^2$ and $Q_x = -y^2$ so $\vec{F}$ isn't conservative. Shucks.

The curve C is nice and has an easy parametrization $\vec{r}(t) = \langle 5\cos t, -5\sin t \rangle$ $0 \le t \le 2\pi$
so maybe I can compute the line integral directly as $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$
But $\vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)$ looks complicated

The region D enclosed by C is nice, so we can try Green's theorem 

Notice that our curve C is oriented negatively and we'll need to introduce a negative sign to fix this

$$\oint_C \vec{F}\cdot d\vec{r} = -\oint_C \vec{F}\cdot d\vec{r} = -\iint_D Q_x - P_y \, dA$$

$$= -\iint_D -y^2 - x^2 \, dA$$

$$= \iint_D x^2 + y^2 \, dA$$

$$= \int_0^{2\pi}\int_0^5 r^2 \cdot r\,dr\,d\theta$$

$$= \frac{5^4}{4}\cdot 2\pi = \boxed{\frac{625\pi}{2}}$$

Remember,
$dA = dy\,dx$ in rectangular
$dA = r\,dr\,d\theta$ in polar

Friday November 13

Reminders

- All Week 12 WebAssign due before midnight on Sunday

13.4 Green's Theorem

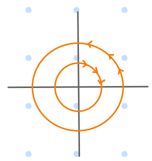
Recap

Let $\vec{F} = \langle P, Q \rangle$ be a vector field, C be a closed curve, and D be the region inside C.

- If $\vec{F}$ is conservative, $\oint_C \vec{F} \cdot d\vec{r} = 0$
- For all $\vec{F}$, $\oint_{C=\partial D} \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$

ex3) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y) = \langle xe^{-2x}, \; x^4 + 2x^2y^2 \rangle$ where C consists of the unit circle oriented clockwise and $x^2+y^2=4$ oriented counterclockwise.

Sketch the curve C



Is $\vec{F}$ conservative? $\quad P_y = 0$, $Q_x = 4x^3 + 4xy^2$ $\quad$ Not conservative

Notice C is the positively oriented boundary of the ring-shaped region D between the circles
Try Green's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$$

$$= \iint_D 4x^3 + 4xy^2 \, dA$$

$$= \iint_D 4x(x^2+y^2) \, dA$$

$$= \int_0^{2\pi} \int_1^2 (4r\cos\theta)(r^2) \, r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} \cos\theta \int_1^2 r^4 \, dr \, d\theta$$

$$= \boxed{0}$$

Heres a sneaky way to compute the area of a region using Green's Theorem!

Recall that $\iint_D 1\,dA$ computes the area of D.

Notice that if we happened to have a vector field $\vec{F}=\langle P,Q\rangle$ such that $Q_x-P_y=1$, then Green's theorem would say

$$\oint_{\partial D}\vec{F}\cdot d\vec{r}=\underbrace{\iint_D 1\,dA}_{\text{Area of } D}$$

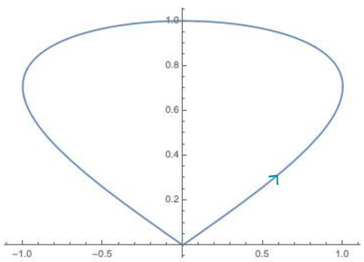
So if we invent a vector field $\vec{F}=\langle P,Q\rangle$ with $Q_x-P_y=1$, we could use Green's theorem to compute area.

ex 4) Find a vector field $\vec{F}=\langle P,Q\rangle$ such that $Q_x-P_y=1$

There are infinitely many possibililies and here are a few

$\vec{F}=\langle 0,x\rangle$ $\quad$ $\vec{F}=\langle -y,0\rangle$ $\quad$ $\vec{F}=\langle -\frac{1}{2}y,\frac{1}{2}x\rangle$

ex 5) Calculate the area inside the curve C given by $\vec{r}(t)=\langle \sin 2t,\sin t\rangle$ for $0\le t\le\pi$, shown below.



Area $=\iint_D 1\,dA = \iint_D Q_x-P_y\,dA$ if $Q_x-P_y=1$

$=\oint_{\partial D}\vec{F}\cdot d\vec{r}$ if $\vec{F}=\langle P,Q\rangle$ satisfies $Q_x-P_y=1$

We'll use $\vec{F}=\langle 0,x\rangle$ here

Compute $\vec{r}'(t)$ $\quad r'(t)=\langle 2\cos 2t,\cos t\rangle$

Compute $\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)$ $\quad \langle 0,\sin 2t\rangle\cdot\langle 2\cos 2t,\cos t\rangle$

$\sin(2t)\cos(t)$

Write integral $\quad \int_0^\pi \sin(2t)\cos(t)\,dt$

$\int_0^\pi 2\sin t\cos t\cos t\,dt$ $\quad u=\cos t$, $du=-\sin t\,dt$

$-\int_1^{-1} 2u^2\,du$

$\boxed{4/3}$

13.5 Curl and Divergence

There are two more kinds of derivatives to learn!

The divergence of a vector field $\vec{F} = \langle P, Q, R \rangle$ is denoted $\nabla \cdot \vec{F}$.
It is calculated via

$$\nabla \cdot F = P_x + Q_y + R_z$$

The curl of a vector field $\vec{F} = \langle P, Q, R \rangle$ is denoted $\nabla \times \vec{F}$
It is calculated via

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

Note: If you haven't learned the formulas for divergence and curl yet, go to Piazza > Tasks for Week 12 > Thursday Nov 12 and use the links to learn and practice the formulas.

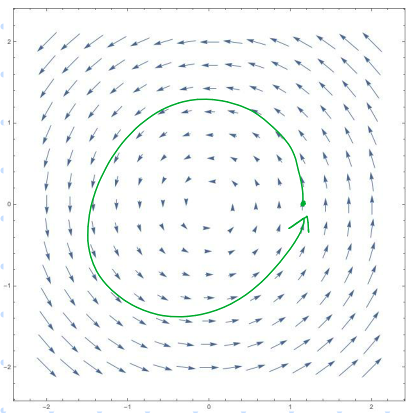
This excellent video will help us get more intuition for what div and curl mean

https://youtu.be/rB83DpBJQsE

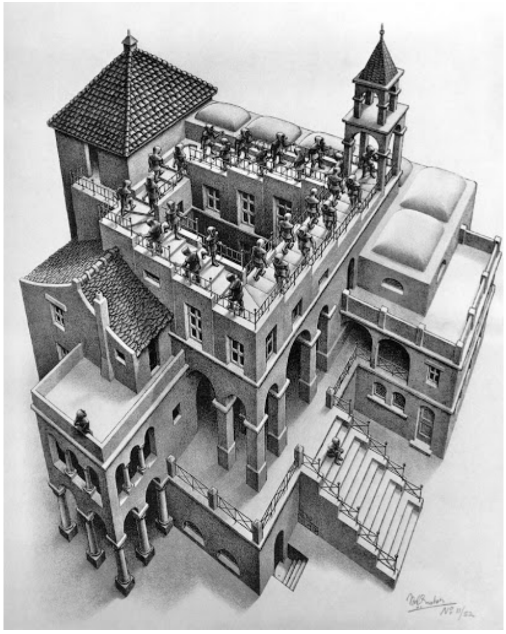
Useful facts about curl and div

- If $\vec{F}$ is conservative, then curl of $\vec{F}$ is zero. In other words, $\boxed{\nabla \times (\nabla f) = \vec{0}}$

  Intuition: If $z = f(x,y)$ is a surface, then ∇f is a vector field where arrows point "uphill" in the direction of greatest increase. That vector field can't have any "swirly" points, since that would mean you can walk on a loop, ending where you started, and somehow be going "uphill" the whole time. That's impossible!



A loop that is "uphill" like this can't occur if $\vec{F} = \nabla f$



Escher defies reality

- The divergence of the curl of $\vec{F}$ is zero. In other words, $\boxed{\nabla \cdot (\nabla \times \vec{F}) = 0}$
- The divergence of the gradient of f, denoted $\boxed{\nabla \cdot \nabla f = \nabla^2 f}$ is called the Laplace operator.
- If curl $\vec{F} = 0$, we say $\vec{F}$ is $\boxed{\text{irrotational}}$
- If div $\vec{F} = 0$, we say $\vec{F}$ is $\boxed{\text{incompressible}}$

We'll use curl and div to upgrade some earlier ideas. Notice that if we turn $\vec{F}=\langle P,Q\rangle$ into a 3D vector field by tacking on a zero in the z-component, then

$$\text{curl}\,\langle P,Q,0\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= (0-Q_z)\vec{i} - (0-P_z)\vec{j} + (Q_x - P_y)\vec{k}$$

P and Q didn't have any z-dependence

$$= (Q_x - P_y)\vec{k}$$

- Test for conservativeness, upgraded

If a vector field ~~$\vec{F}=\langle P,Q\rangle$~~ $\vec{F}=\langle P,Q,R\rangle$ is conservative, ~~then $Q_x = P_y$~~ then $\text{curl}\,\vec{F}=0$ (old version) (upgrade!)

If ~~$Q_x = P_y$~~ $\text{curl}\,\vec{F}$ on a simply-connected domain, then $\vec{F}$ is conservative.

- Green's Theorem, upgraded

$$\oint_{\partial D} \vec{F}\cdot d\vec{r} = \cancel{\iint_D Q_x - P_y \, dA} \;\; \iint_D \text{curl}\,\vec{F}\cdot\vec{k}\, dA$$

Ok this one is not really an upgrade since it's actually more work to compute curl $\vec{F}$. Honest upgrade coming soon.

Monday November 16

Reminders

- WebAssign 13.5
- HW 13 section 13.5 and A1
- Study for Check-in 20
  - Be able to state Green's theorem
  - Redo examples from 13.4 notes

13.5 Curl and divergence (cont)

Recap

The curl of a vector field $\vec{F} = \langle P, Q, R \rangle$ is $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

The divergence of a vector field $\vec{F} = \langle P, Q, R \rangle$ is $\nabla \cdot \vec{F} = P_x + Q_y + R_z$

ex 1) Is there a vector field $\vec{G}$ on $\mathbb{R}^3$ such that $\text{curl}\,\vec{G} = \langle x\sin y, \cos y, z - xy \rangle$?

We know that for any vector field $\vec{F}$, it is always true that $\text{div}(\text{curl}\,\vec{F}) = 0$. So if $\langle x\sin y, \cos y, z-xy \rangle$ were the curl of some other vector field $\vec{G}$, then its divergence should be zero.

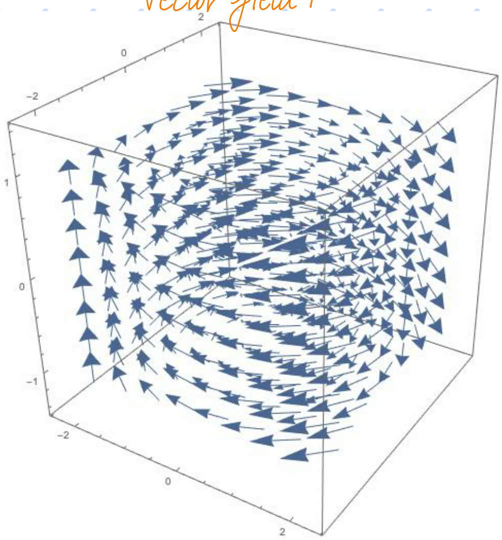
$$\begin{aligned} \text{div}(\langle x\sin y, \cos y, z-xy \rangle) &= \nabla \cdot \langle x\sin y, \cos y, z-xy \rangle \\ &= \sin y - \sin y + 1 \\ &= 1 \neq 0 \end{aligned}$$

No this vector field is not the curl of some other vector field.

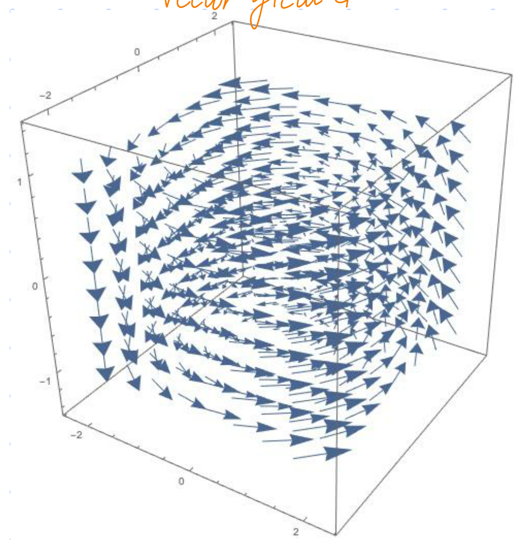
ex 2) Compute the curl of $\vec{F} = \langle y, -x, 0 \rangle$ and $\vec{G} = \langle -y, x, 0 \rangle$. What's the difference?

$$\text{curl}\,\vec{F} = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \langle 0, 0, -2 \rangle \qquad \text{curl}\,\vec{G} = \nabla \times G = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

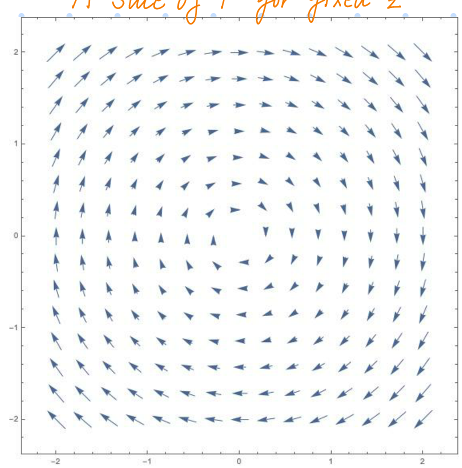
Vector field $\vec{F}$



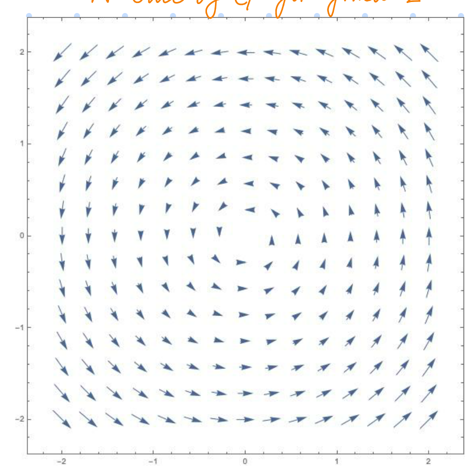
Vector field $\vec{G}$



A slice of $\vec{F}$ for fixed z



A slice of $\vec{G}$ for fixed z



I notice that when the vector field describes a circular flow, the curl changes sign when the flow changes direction.

Fact: The curl of $\vec{F}$ is a vector whose direction is governed by the right-hand rule.
Let the fingers of your right hand follow the rotation motion described by the vector field.
Then your thumb will point in the direction of the curl vector.

See animations at mathinsight.org/curl_subtleties ex3) (from textbook)

12. Let f be a scalar field and $\mathbf{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

a. curl f — nonsense

b. grad f — vector field

c. div $\mathbf{F}$ — scalar field

d. curl(grad f) — vector field

e. grad $\mathbf{F}$ — nonsense

f. grad(div $\mathbf{F}$) — vector field

g. div(grad f)

h. grad(div f)

i. curl(curl $\mathbf{F}$)

j. div(div $\mathbf{F}$)

k. (grad f) $\times$ (div $\mathbf{F}$)

l. div(curl(grad f))

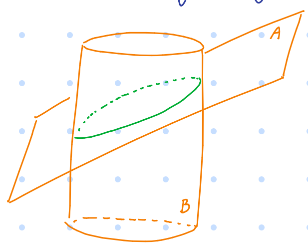
on your own :) (in WebAssign 13.5a)

Mixed problems (since Quiz 8 is cumulative)

Let A be the surface $z=x+y$ and B be the surface $x^2+y^2=1$

1.) Write an integral that computes the arc length of the intersection of A and B

The general shape is a cylinder cut by a plane, so the projection onto the xy-plane is the unit circle.



$\vec{r}(t) = \langle \cos t, \sin t, \sin t + \cos t \rangle, \quad 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, \cos t - \sin t \rangle$

$|\vec{r}\,'(t)| = (\sin^2 t + \cos^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t)^{1/2}$

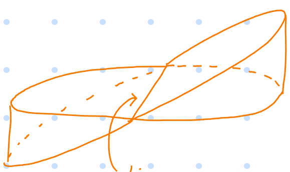
$= (2 + 2\sin t \cos t)^{1/2}$

Arc length $= \int 1\, ds$

$= \int_a^b |\vec{r}\,'(t)|\, dt$

$$\boxed{\text{Arc length} = \int_0^{2\pi} \sqrt{2 - 2\sin t\cos t}\; dt}$$

2.) Write an integral that computes the surface area of the part of B below A and above $z=0$



$\vec{r}(u,v) = \langle \cos u, \sin u, v \rangle \quad -\frac{\pi}{4} \le u \le \frac{3\pi}{4}, \ 0 \le v \le \sin u + \cos u$

$\vec{r}_u = \langle -\sin u, \cos u, 0 \rangle$

$\vec{r}_v = \langle 0, 0, 1 \rangle$

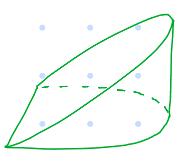
$|\vec{r}_u \times \vec{r}_v| = 1$

Surface area $= \iint_D 1\, dS$

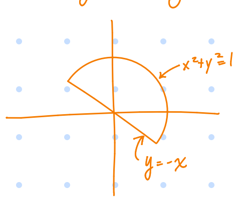
$= \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$

$$\boxed{\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\cos u + \sin u} 1 \; dv\,du}$$

3.) Write an integral that computes the volume of the solid inside B, below A and above $z=0$



Domain of integration



Volume under a surface $= \iint_D f(x,y)\, dA$

$= \iint_D x+y \; dA$

$$\boxed{= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^1 (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta}$$

4.) Write an integral that computes the work done by $\vec{F} = \langle 3y + 3x^2y, 3x + x^3 \rangle$ on a particle traveling the curve in problem 1 in the counterclockwise direction.

$\vec{F}$ is conservative and the path is a simple closed curve, so the work is $\boxed{\text{zero}}$!

5.) Find the minimum value of $f(x,y) = 4xy$ subject to the constraint B.

$\nabla f = \langle 4y, 4x \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 2y = \lambda x \\ 2x = \lambda y \\ x^2 + y^2 = 1 \end{cases}$$

Crit pts: $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2$

$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -2$

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -2$

$\boxed{\text{min value} = -2}$

Tuesday November 17

Reminders

- Review how to compute surface area (section 12.6)
  $\iint_D |\vec{r}_u \times \vec{r}_v| \, dA$ (more important)
  $\iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dA$ (less important)

Summary of facts about $\int_C \vec{F} \cdot d\vec{r}$ and vector fields

If $\vec{F}$ is conservative

... curl $\vec{F} = \vec{0}$

... $\vec{F}$ has a potential function f such that $\nabla f = \vec{F}$

... $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

... $\int_C \vec{F} \cdot d\vec{r}$ is path-independent

... $\oint_C \vec{F} \cdot d\vec{r} = 0$ for a simple closed curve C

If curl $\vec{F} = \vec{0}$ on a simply connected region, then $\vec{F}$ is conservative.

In all circumstances

... curl$(\nabla f) = \vec{0}$

... div(curl $\vec{F}$) $= 0$

... $\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$ (Green's Theorem)

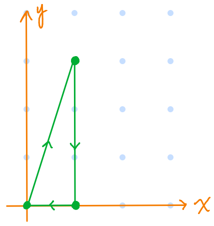
... $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$

Go to student.desmos.com and use code QDC AC8
Do the "Conservative or not" activity

Check-in 20

Use Green's theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle \sqrt{1+x^3},\, 2xy \rangle$ and C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ oriented clockwise

Draw C



Is $\vec{F}$ conservative?

$Q = 2xy \qquad P = \sqrt{1+x^2}$

$Q_x = 2y \qquad P_y = 0$

No, not conservative, so we can't conclude $\oint_C \vec{F}\cdot d\vec{r} = 0$

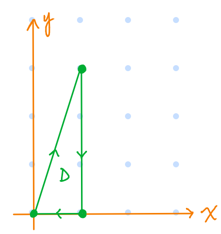
Use Green's theorem

Our curve C has the negative orientation relative to D, so we must correct with a negative sign

$$\oint_C \vec{F}\cdot d\vec{r} = -\iint_D Q_x - P_y \, dA$$

$$= -\int_0^1 \int_0^{3x} 2y \; dy\,dx$$

$$= \boxed{-3}$$

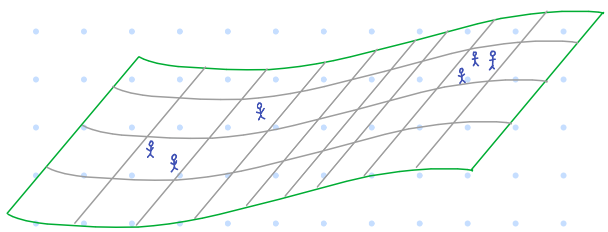


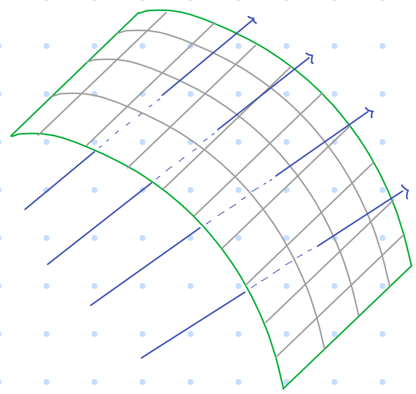
Wednesday November 18

- Compile HW 13 for André

13.6 Surface integrals

A surface integral is a double integral where the domain of integration is some surface S. There are two different kinds - one where we integrate a scalar function and one where we integrate a vector function





Example

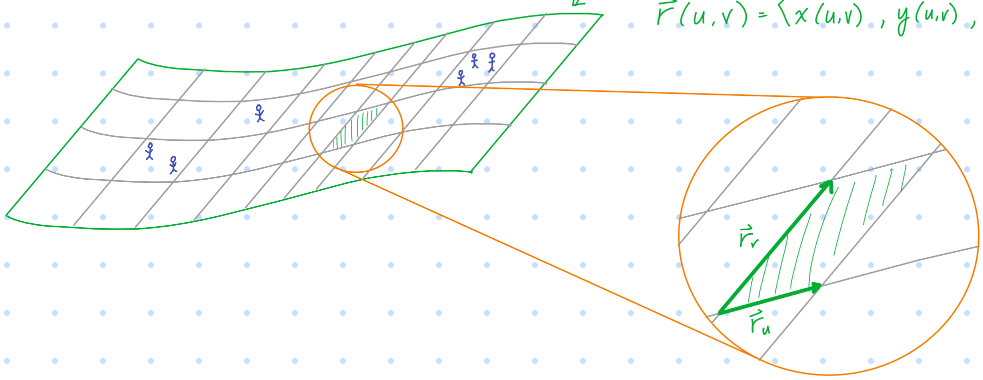
Let $f(x,y,z)$ be the population density at point (x,y,z). Then the integral of $f(x,y,z)$ over a surface computes the total population on the surface

Example

Let $\vec{F}(x,y,z)$ be a vector field depicting the rate of flow of some fluid. Then the integral of $\vec{F}$ over the surface gives the rate of flow of the fluid through the surface.

Surface integral of scalar function - intuition

Surface S with parametrization $\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$



Area of tiny parallelogram patch of surface S is

$$|\vec{r}_u \times \vec{r}_v|$$

Surface integral of scalar function - computation

$$\iint_S f(x,y,z)\, dS = \iint_{u,v \in D} f(\vec{r}(u,v)) \underbrace{|\vec{r}_u \times \vec{r}_v|\, dA}_{dS\text{, a tiny patch of } S}$$

If the surface S happens to be of the form $z = g(x,y)$, then

$$\iint_S f(x,y,z)\, dS = \iint_{u,v \in D} f(u,v,g(u,v)) \underbrace{\sqrt{z_x^2 + z_y^2 + 1}\, dA}_{dS\text{, a tiny patch of } S}$$

ex 1) Evaluate $\iint_S x^2 \, dS$ where S is the sphere $x^2+y^2+z^2=9$

Parametrize S — Use spherical-inspired parametrization with $\rho=3$, θ and ϕ varying

$$\vec{r}(\theta,\phi) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle \quad 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi$$

Compute $|\vec{r}_\theta \times \vec{r}_\phi|$

$$\vec{r}_\theta = \langle -3\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0 \rangle$$

$$\vec{r}_\phi = \langle 3\cos\phi\cos\theta, 3\cos\phi\sin\theta, -3\sin\phi \rangle$$

$$\vec{r}_\theta \times \vec{r}_\phi = \langle -9\sin^2\phi\cos\theta, -9\sin^2\phi\sin\theta, -9\sin\phi\cos\phi\cos^2\theta - 9\sin\phi\cos\phi\sin^2\theta \rangle$$

$$= \langle -9\sin^2\phi\cos\theta, -9\sin^2\phi\sin\theta, -9\sin\phi\cos\phi \rangle$$

$$|\vec{r}_\theta \times \vec{r}_\phi| = (81\sin^4\phi\cos^2\theta + 81\sin^4\phi\sin^2\theta + 81\sin^2\phi\cos^2\phi)^{1/2}$$

$$= (81\sin^4\phi + 81\sin^2\phi\cos^2\phi)^{1/2}$$

$$= (81\sin^2\phi(\sin^2\phi + \cos^2\phi))^{1/2}$$

$$= 9\sin\phi$$

Write integral

$$\int_0^{\pi}\int_0^{2\pi} \underbrace{(3\sin\phi\cos\theta)}_{x}{}^2 \underbrace{(9\sin\phi)}_{|\vec{r}_\theta \times \vec{r}_\phi|} \, d\theta \, d\phi$$

$$81 \int_0^{\pi} \sin^3\phi \int_0^{2\pi} \cos^2\theta \, d\theta \, d\phi$$

$$81\pi \int_0^{\pi} \sin^3\phi \, d\phi$$

$$81\pi \int_0^{\pi} \sin\phi(1-\cos^2\phi)\, d\phi \qquad u=\cos\phi \quad du=-\sin\phi\, d\phi$$

$$-81\pi \int_1^{-1} 1-u^2 \, du$$

$$-81\pi \left[u - \tfrac{1}{3}u^3\right]_1^{-1}$$

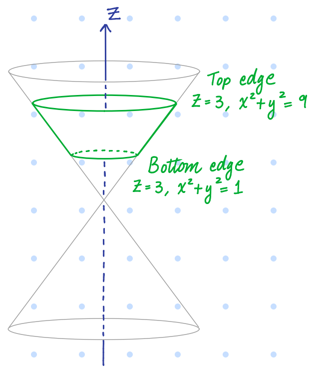
$$-81\pi \left[-1+\tfrac{1}{3}-1+\tfrac{1}{3}\right]$$

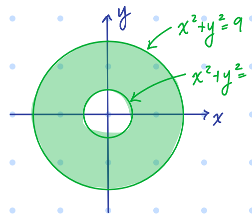
$$27 \cdot 4\pi$$

$$\boxed{108\pi}$$

ex 2) Evaluate $\iint_S x^2 z^2 \, dS$ when S is the part of the cone $z^2 = x^2 + y^2$ between $z=1$ and $z=3$

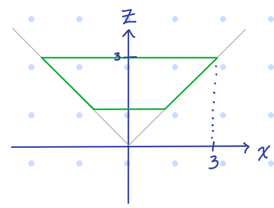
Sketch and parametrize S





Use the fact that $z = \sqrt{x^2+y^2}$ is a function of x and y

$\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$



$\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$

Compute $|\vec{r}_u \times \vec{r}_v|$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 + 1}$$

$$= \sqrt{\left(\frac{u}{\sqrt{u^2+v^2}}\right)^2 + \left(\frac{v}{\sqrt{u^2+v^2}}\right)^2 + 1}$$

$$= \sqrt{\frac{u^2+v^2}{u^2+v^2} + 1}$$

$$= \sqrt{2}$$

Write integral

$$\iint_{u,v \in D} \underbrace{u^2(u^2+v^2)}_{x^2 z^2} \underbrace{(\sqrt{2})}_{|\vec{r}_u \times \vec{r}_v|} \, dA$$

$$\sqrt{2} \int_0^{2\pi} \int_1^3 \underbrace{r^2 \cos^2\theta \cdot r^2}_{x^2 z^2} \cdot \underbrace{r \, dr \, d\theta}_{dA \text{ in polar}}$$

$$\sqrt{2} \int_0^{2\pi} \int_1^3 r^5 \cos^2\theta \, dr \, d\theta$$

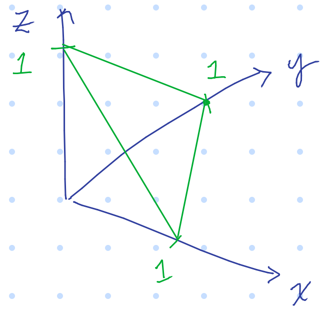
$$\sqrt{2} \int_0^{2\pi} \cos^2\theta \int_1^3 r^5 \, dr \, d\theta$$

$$\sqrt{2} \; \pi \; \tfrac{1}{6}(3^6 - 1)$$

$$\boxed{\frac{364\pi\sqrt{2}}{3}}$$

ex 3) Evaluate $\iint_S yz\, dS$ where S is the part of the plane $x+y+z=1$ in the first octant.

Sketch and parameterize S

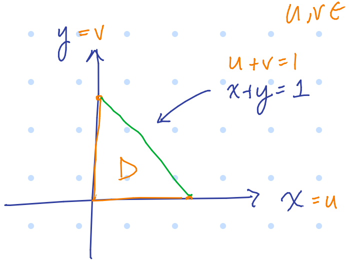


$z = 1 - x - y$

$\vec{r}(u,v) = \langle u, v, 1-u-v \rangle$

Compute $|\vec{r}_u \times \vec{r}_v| = \sqrt{(-1)^2 + (-1)^2 + 1} = \sqrt{3}$

Write integral $\displaystyle\iint_{u,v \in D} v(1-u-v)\sqrt{3}\, dA$



$$\sqrt{3}\int_0^1\int_0^{1-v} v(1-u-v)\, du\, dv$$

$$\sqrt{3}\int_0^1\int_0^{1-v} v - uv - v^2\, du\, dv$$

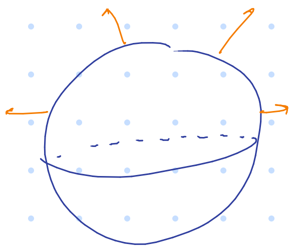
$$\sqrt{3}\int_0^1 \left[uv - \tfrac{1}{2}u^2 v - v^2 u\right]_0^{1-v} dv$$

$$\sqrt{3}\int_0^1 (1-v)v - \tfrac{1}{2}(1-v)^2 v - v^2(1-v)\, dv$$

$$\boxed{\sqrt{3}/24}$$

A surface has 2 possible orientations, if it is orientable at all.

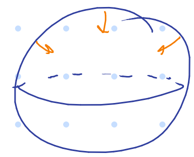
example: sphere



normal vectors to the surface of sphere

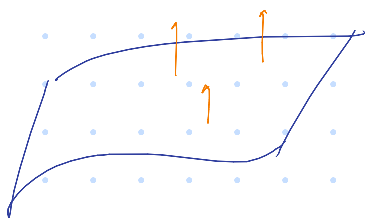
oriented "outward"

or

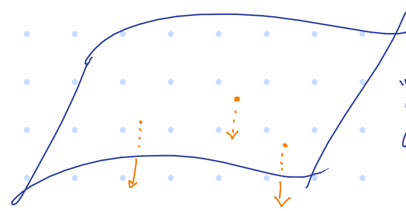


oriented "inward"

example: non-closed surface



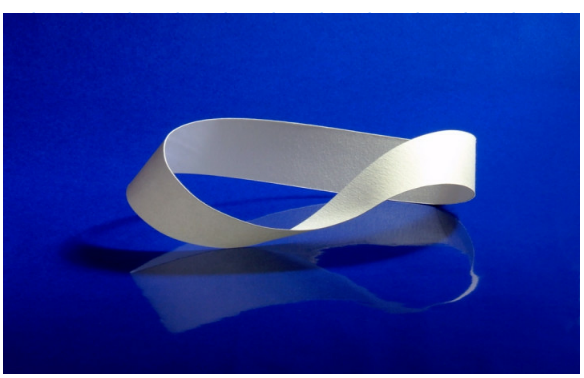
"upward" orientation



"downward" orientation

Some objects are <u>not</u> orientable at all

A Möbius strip does not have distinct "back" and "front"



A Klein bottle does not have distinct "inside" and "outside"

Friday November 20

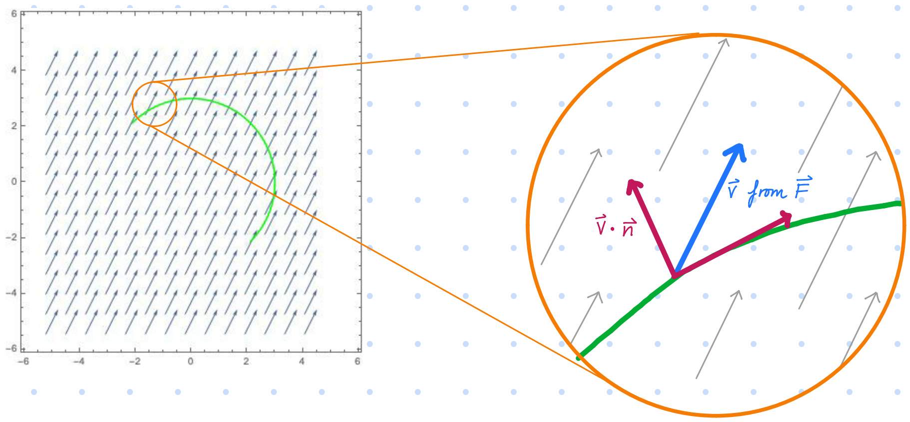
Reminders

- All Week 13 WebAssign due Sunday before midnight
- HW 14 section 13.6 (relevant to Quiz 7)
- Study for Quizzes 7 and 8

13.6 Surface integrals (cont)

Recap

From yesterday's project: If we think of the arrows of vector field $\vec{F}$ as representing the velocity of a flowing liquid, then we can calculate the flux (flow rate) through an object.

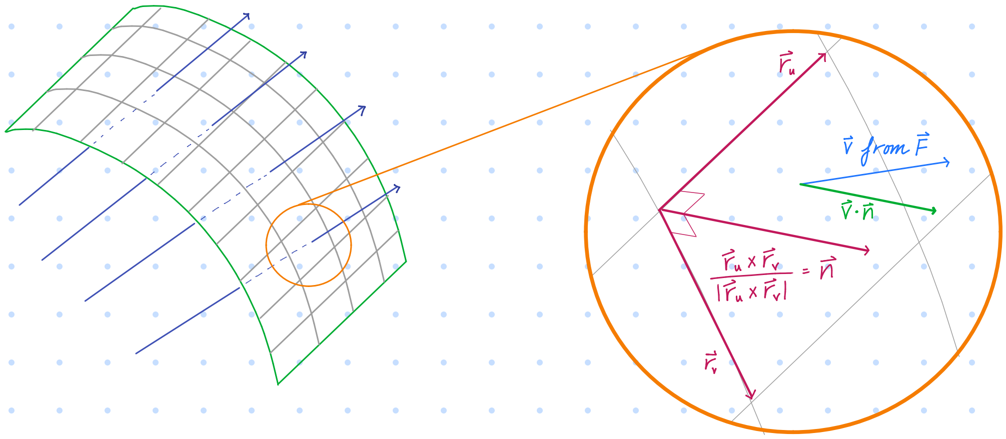


The component of $\vec{v}$ that actually passes through C is the orthogonal component $\vec{v}\cdot\vec{n}$. Then the flux across all of C is

$$\int_C \vec{F}\cdot\vec{n}\, ds$$

But real-life filters are surfaces in 3D space, not just a flat curve in 2D space! So let's develop the 3D version.

Flux - the intuition



The unit normal to the surface is $\dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \vec{n}$

The component of $\vec{F}$ orthogonal to S is $\vec{F}\cdot\vec{n}$

Then the total flux across the surface S is $\iint_S \vec{F}\cdot\vec{n}\, dS$

Flux - the computation

We need to decide which side of S is the "outside"

Flux of $\vec{F}$ across S with some orientation

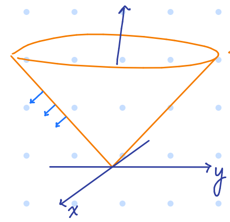
$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_{u,v\in D} \left(\vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \right) \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA$$

These scalars cancel!

$$\iint_{u,v\in D} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

You may need to replace the vector $\vec{r}_u \times \vec{r}_v$ with its opposite $-(\vec{r}_u \times \vec{r}_v)$ in order to match your desired choice of "outward" direction ex 4) Compute the flux of $\vec{F} = \langle x, y, z^4 \rangle$ across the surface S, where S is the part of the cone $z = \sqrt{x^2+y^2}$ beneath $z=1$ with downward orientation

Sketch and parametrize S



$\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$

u, v in $u^2+v^2 \leq 1$

Compute $\vec{r}_u \times \vec{r}_v$ and verify orientation

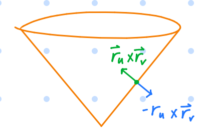
$$\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$$

$$\vec{r}_u = \left\langle 1, 0, \frac{u}{\sqrt{u^2+v^2}} \right\rangle$$

$$\vec{r}_v = \left\langle 0, 1, \frac{v}{\sqrt{u^2+v^2}} \right\rangle$$

$$\vec{r}_u \times \vec{r}_v = \left\langle \frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1 \right\rangle$$

Notice the z-component is positive! So this normal vector must point upward, opposite to the direction we want! Introduce a negative sign to fix this



our normal : $\left\langle \frac{u}{\sqrt{u^2+v^2}}, \frac{v}{\sqrt{u^2+v^2}}, -1 \right\rangle$

Compute $\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)$

$$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = \langle u, v, (u^2+v^2)^2 \rangle \cdot \left\langle \frac{u}{\sqrt{u^2+v^2}}, \frac{v}{\sqrt{u^2+v^2}}, -1 \right\rangle$$

$$= \frac{u^2}{\sqrt{u^2+v^2}} + \frac{v^2}{\sqrt{u^2+v^2}} - (u^2+v^2)^2$$

$$= \sqrt{u^2+v^2} - (u^2+v^2)^2$$

Write integral and compute

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{u,v \in D} \sqrt{u^2+v^2} - (u^2+v^2)^2 \, dA$$

$$= \int_0^{2\pi} \int_0^1 (r - r^4)\, r \, dr \, d\theta$$

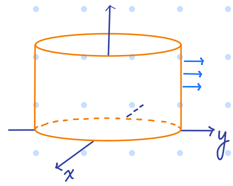
$$= 2\pi \int_0^1 r^2 - r^5 \, dr$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{6} \right)$$

$$= \boxed{\frac{\pi}{3}}$$

ex 5) A fluid has density 870 kg/m^3 and flows with velocity $\vec{V} = z\,\vec{i} + y^2\,\vec{j} + x^2\,\vec{k}$ where x, y, z are measured in meters and the components of $\vec{V}$ are in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \le z \le 1$.

Sketch and parametrize S



$\vec{r}(u,v) = \langle 2\cos u, 2\sin u, v \rangle$

$0 \le u \le 2\pi$, $0 \le v \le 1$

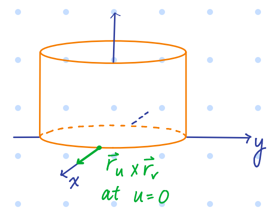
Compute $\vec{r}_u \times \vec{r}_v$ and verify orientation

$\vec{r}(u,v) = \langle 2\cos u, 2\sin u, v \rangle$

$\vec{r}_u = \langle -2\sin u, 2\cos u, 0 \rangle$
$\vec{r}_v = \langle 0, 0, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2\cos u, 2\sin u, 0 \rangle$

Notice if $u=0$, then $\vec{r}_u \times \vec{r}_v$ is the vector $\langle 2,0,0 \rangle$. This is indeed the "outward" direction. No correction needed



Compute $\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)$

$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = \langle v, 4\sin^2 u, 4\cos^2 u \rangle \cdot \langle 2\cos u, 2\sin u, 0 \rangle$

$= 2v\cos u + 8\sin^3 u$

Write integral and compute

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^{2\pi} v\cos u + 4\sin^3 u \; du\, dv$$

$$= \int_0^1 v \int_0^{2\pi} \cos u\, du\, dv + \int_0^1 \int_0^{2\pi} 4\sin^3 u\, du\, dv$$

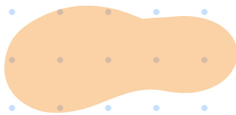
$$= 0 \quad m^3/s$$

Flow = flux · density = $(0)(870)$ kg/s $= \boxed{0 \text{ kg/s}}$

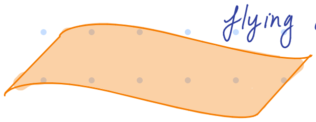
An aside about boundaries

What is the boundary of each object?

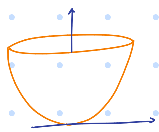
(a)



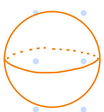
(b) flying carpet



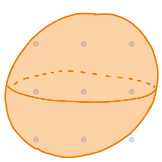
(c) paraboloid cup (shell)



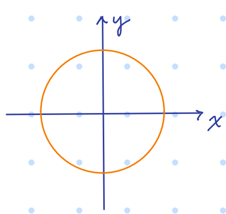
(d) unit sphere (a shell, not filled)



(e) solid ball (filled)

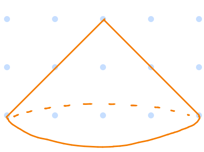


(g) unit circle (not filled)



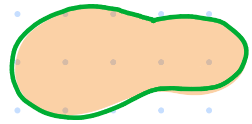
(h) line segment

(i) solid cone (filled inside)

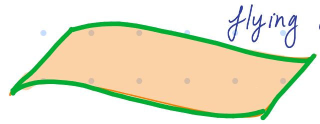


Answers in green

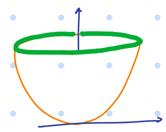
(a)



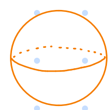
(b) flying carpet



(c) paraboloid cup (shell)

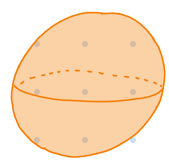


(d) unit sphere (a shell, not filled)

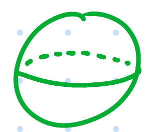


none

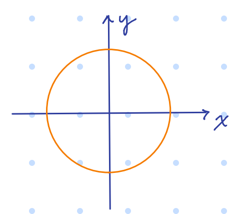
(e) solid ball (filled)



shell



(g) unit circle (not filled)

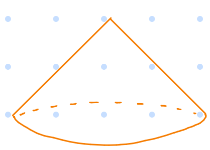


none

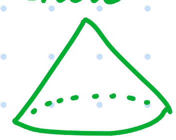
(h) line segment

two points

(i) solid cone (filled inside)

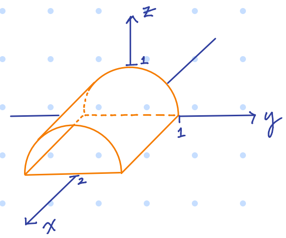


shell



ex 6) Compute the flux of $\vec{F} = \langle x^2, y^2, z^2 \rangle$ across the surface S, where S is the boundary of the solid half-cylinder $0 \le z \le \sqrt{1-y^2}$, $0 \le x \le 2$. Use outward orientation

Sketch and parametrize S



$\langle 2, u\cos v, u\sin v \rangle \quad 0 \le u \le 1 \quad 0 \le v \le \pi$

$\langle 0, u\cos v, u\sin v \rangle \quad 0 \le u \le 1 \quad 0 \le v \le \pi$

Four pieces form the boundary!

Top	$\vec{r}_1$	$= \langle u, \cos v, \sin v \rangle$	$0 \le u \le 2$	$0 \le v \le \pi$
Bottom	$\vec{r}_2$	$= \langle u, v, 0 \rangle$	$0 \le u \le 2$	$-1 \le v \le 1$
Front	$\vec{r}_3$	$= \langle 2, u, v \rangle$	$-1 \le u \le 1$	$0 \le v \le \sqrt{1-y^2}$
Back	$\vec{r}_4$	$= \langle 0, u, v \rangle$	$-1 \le u \le 1$	$0 \le v \le \sqrt{1-y^2}$

Note: I could use polar-inspired coordinates for Front and Back, but I chose rectangular because it makes the normal vector easy to find.

Compute $r_u \times r_v$ and verify orientation

For top piece $\vec{r}_1$

$\vec{r}_1 = \langle u, \cos v, \sin v \rangle$

$(\vec{r}_1)_u = \langle 1, 0, 0 \rangle$

$(\vec{r}_1)_v = \langle 0, -\sin v, \cos v \rangle$

$(\vec{r}_1)_u \times (\vec{r}_1)_v = \langle 0, -\cos v, -\sin v \rangle$

but the orientation is wrong

Our normal vector: $\boxed{\langle 0, \cos v, \sin v \rangle}$

For bottom piece $\vec{r}_2$

$\vec{r}_2 = \langle u, v, 0 \rangle$

Useful tip: This piece of the boundary is a flat rectangle, so we can find the normal vector visually. It should be the downward unit vector $\boxed{\langle 0, 0, -1 \rangle}$

For front piece $\vec{r}_3$ and back piece $\vec{r}_4$

$\vec{r}_3 = \langle 2, u, v \rangle$

$\vec{r}_4 = \langle 0, u, v \rangle$

Useful tip: These pieces are flat, so their normal vectors can be found visually

$\boxed{\begin{array}{l} \vec{n} \text{ for front: } \langle 1, 0, 0 \rangle \\ \vec{n} \text{ for back: } \langle -1, 0, 0 \rangle \end{array}}$

$\vec{r}_3$ should have a normal that points forward while $\vec{r}_4$ should point backward

Compute $\vec{F}(\vec{r}(u,v)) \cdot (r_u \times r_v)$

For top piece $\vec{r}_1$

$\langle u^2, \cos^2 v, \sin^2 v \rangle \cdot \langle 0, \cos v, \sin v \rangle$

$\boxed{\cos^3 v + \sin^3 v}$

For bottom piece $\vec{r}_2$

$\langle u^2, v^2, 0^2 \rangle \cdot \langle 0, 0, -1 \rangle$

$\boxed{0}$

For front piece $\vec{r}_3$

$\langle 2^2, u^2, v^2 \rangle \cdot \langle 1, 0, 0 \rangle$

$\boxed{4}$

For back piece $\vec{r}_4$

$\langle 0, u^2, v^2 \rangle \cdot \langle -1, 0, 0 \rangle$

$\boxed{0}$

Write integral and evaluate

For top piece $\vec{r}_1$

$$\int_0^{\pi} \int_0^2 \cos^3 v + \sin^3 v \; du\, dv$$

$$2 \int_0^{\pi} \cos^3 v + \sin^3 v \; du\, dv$$

$$\boxed{\frac{8}{3}}$$

For bottom piece $\vec{r}_2$

$$\iint 0 \, dA = \boxed{0}$$

For front piece $\vec{r}_3$

$$\iint_{u,v} 4 \, dA = 4 \, (\text{area of } D) = \boxed{2\pi}$$

For back piece $\vec{r}_4$

$$\iint 0 \, dA = \boxed{0}$$

$$\text{Flux} = \frac{8}{3} + 0 + 2\pi + 0 = \boxed{\frac{8}{3} + 2\pi}$$

Check-in 21

Evaluate $\iint_S (x+y+z)\,dS$ where S is a parallelogram given by $x=u+v$, $y=u-v$, $z=1+2u+v$, $0\le u\le 2$, $0\le v\le 1$

(final answer should be $11\sqrt{14}$)

Monday November 23

Reminders

- Study for Quizzes 7 and 8

23rd 13.7 - Stokes' Theorem	24th Review QUIZ 7 QUIZ 8	25th Remote office hours during class time	26th Fall Break No Class	27th Fall Break No Class	WebAssign 13.6b due Sun Nov 29
30th 13.7 (cont.) Corrections for Quizzes 7+8 due	Dec 1st 13.8 - Divergence Theorem	2nd 13.8 (cont.)	3rd P: What is This Thing? #3 (Types of Integrals) HW 14 due	4th A: Fundamental Theorem Practice A: Fundamental Theorem Matching	

13.7 Stokes' Theorem

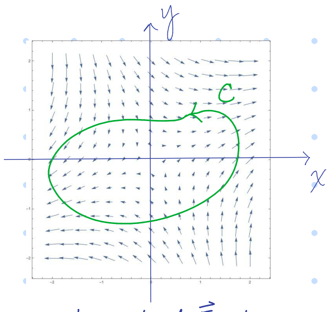
Recall that Green's theorem says for $\vec{F} = \langle P, Q \rangle$, we know $\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$

Stokes' theorem is a souped up version of Green's theorem that works for $\vec{F} = \langle P, Q, R \rangle$

> Stokes' theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then
>
> $$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

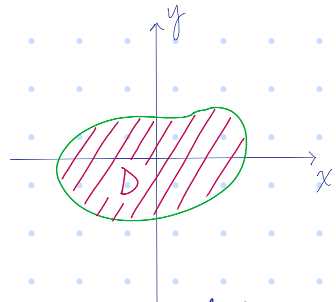
In what way are Green's and Stokes' theorems related?

Green's says these integrals are equal



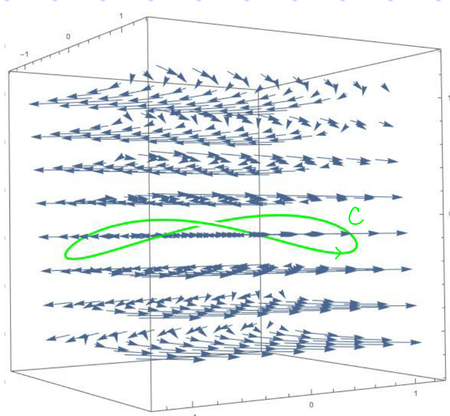
Integral of $\vec{F}$ along the curve C

=



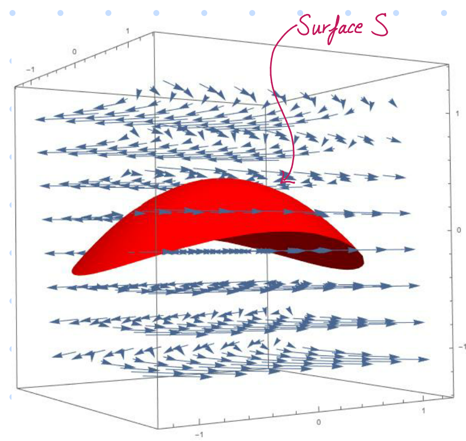
Integral of $Q_x - P_y$ over region D

Stokes' says these integrals are equal



Integral of $\vec{F}$ along the curve C

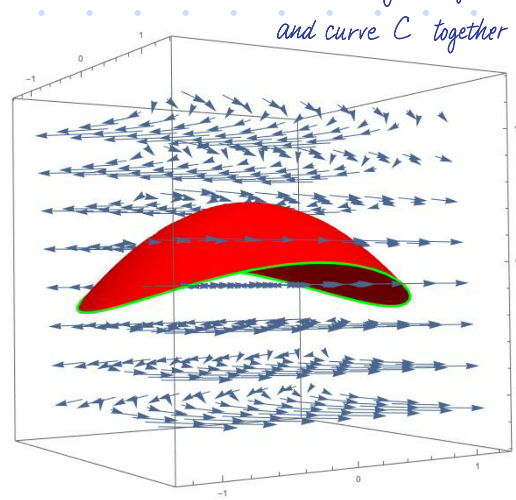
=



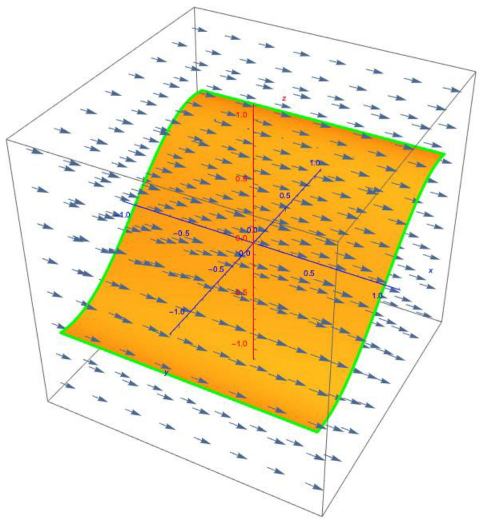
Integral of curl $\vec{F}$ over surface S

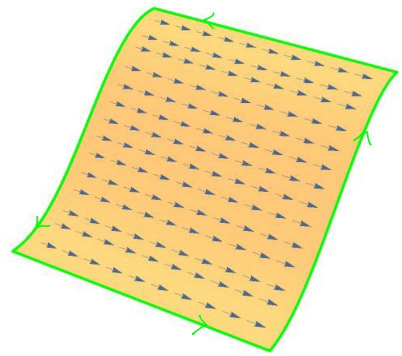
Green's theorem is precisely Stokes' theorem for flat surfaces!

Picture of surface S and curve C together



Stokes' theorem intuition

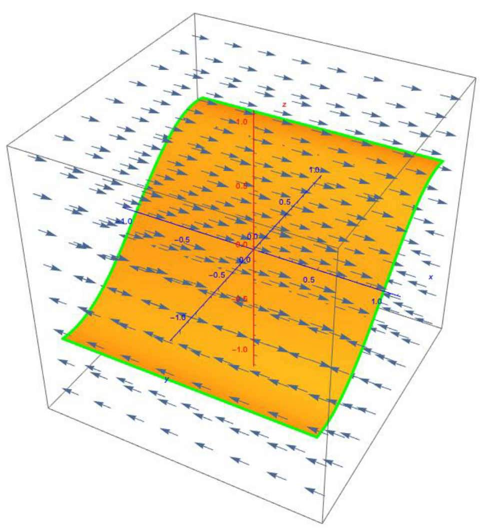


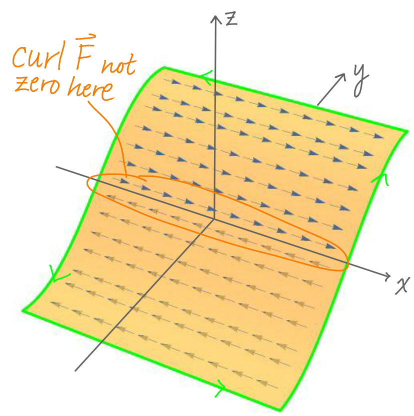


By inspection,

$\int_C \vec{F} \cdot dr = 0$

curl $\vec{F} = \vec{0}$ everywhere

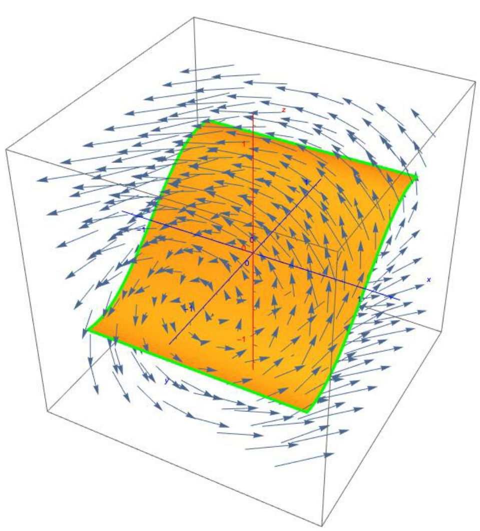


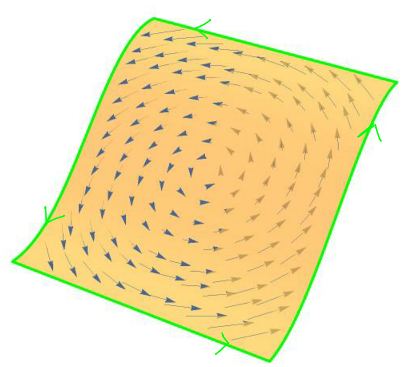


By inspection,

$\int_C \vec{F} \cdot dr =$ negative

curl $\vec{F} = \vec{0}$ most everywhere but not zero (with clockwise motion) along $y=0$



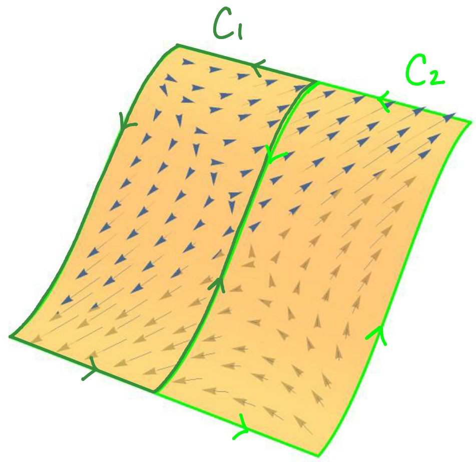


By inspection,

$\int_C \vec{F} \cdot dr =$ positive

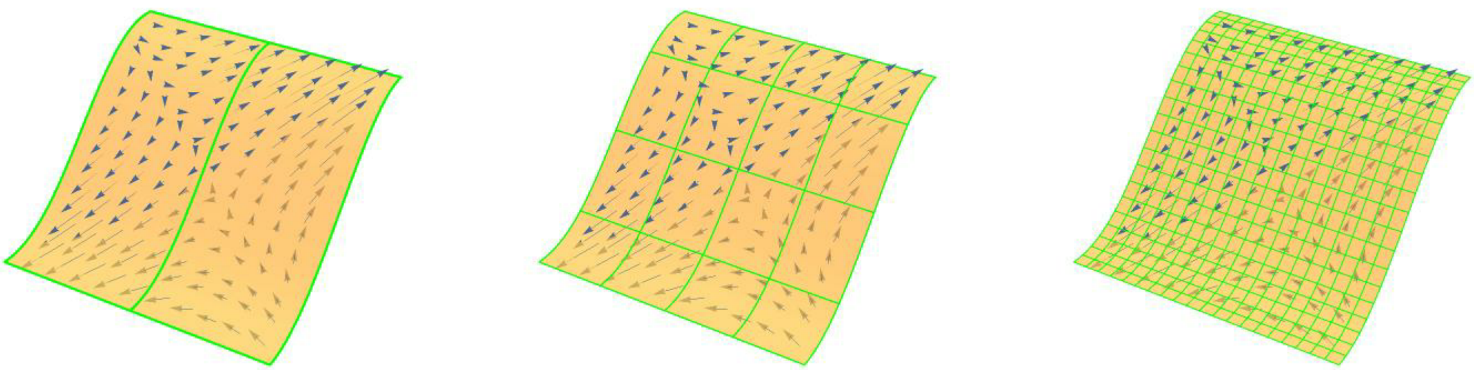
curl $\vec{F} =$ not zero, with counterclockwise motion

If we compute $\int_C \vec{F}\cdot d\vec{r}$ as $\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$, we see that one piece of the integral along C_1 cancels with one piece of the integral along C_2 to form the integral over C.



The same principle applies for further subdivisions

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \cdots + \int_{C_n} \vec{F}\cdot d\vec{r}$$



the curl of $\vec{F}$ is exactly

Well, as the loops get tinier (take a limit), each $\int_{C_k} \vec{F}\cdot d\vec{r}$ produces the component of the curl of $\vec{F}$ orthogonal to the surface. In other words, $\int_{C_k} \vec{F}\cdot d\vec{r} \simeq \vec{F}\cdot\vec{n}$ inside tiny loop C_k. Then adding up every tiny loop across all of S, we get

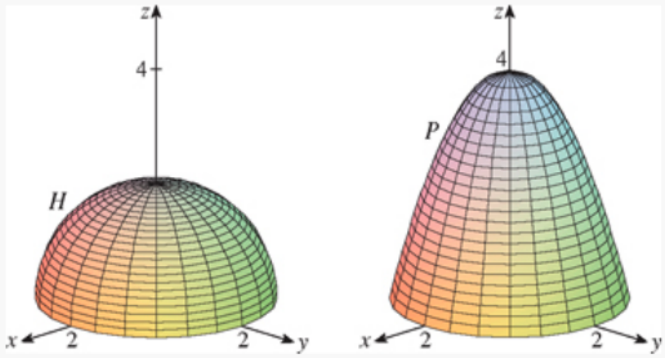
$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \cdots + \int_{C_n} \vec{F}\cdot d\vec{r} \\ &= \underset{\text{in } C_1}{\operatorname{curl}\vec{F}\cdot\vec{n}} + \underset{\text{in } C_2}{\operatorname{curl}\vec{F}\cdot\vec{n}} + \cdots + \underset{\text{in } C_n}{\operatorname{curl}\vec{F}\cdot\vec{n}} \\ &= \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, dS\end{aligned}$$

smbc-comics.com/comic/2014-02-24

ex 1) (part of HW 14)

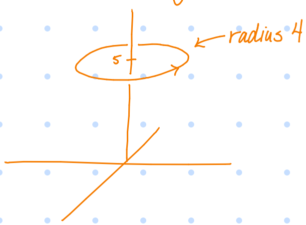
1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_P \text{curl } \mathbf{F} \cdot d\mathbf{S}$$



ex 2) Use Stokes' theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$. Let C be oriented counterclockwise when viewed from above.
$\vec{F}(x,y,z) = \langle yz, 2xz, e^{xy} \rangle$ and C is the circle $x^2+y^2=16$, $z=5$

Sketch of C

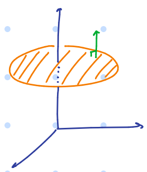


To use Stokes' theorem, we want to create a surface S whose boundary is C. One option is the flat disk inside C oriented in the upward direction

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$ for S the flat disk inside C.

Parameterize S: $\vec{r}(u,v) = \langle u, v, 5 \rangle$ for $u^2+v^2 \le 4$

Find $\vec{n}$:



By inspecti , $\vec{n}$ is the unit vector $\langle 0,0,1 \rangle$

Compute curl $\vec{F}$:

$$\text{Curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 2xz & e^{xy} \end{vmatrix} = \left\langle xe^{xy} - 2x, -(ye^{xy} - y), 2z - z \right\rangle$$

Write integral: $\iint_{u,v} \vec{F}(\vec{r}(u,v)) \cdot \langle 0,0,1 \rangle \, dA$

$$\iint_{u,v} 5 \, dA$$

$$5 \iint_{u,v} dA$$

$$5 \,(\text{area of domain of } u,v)$$

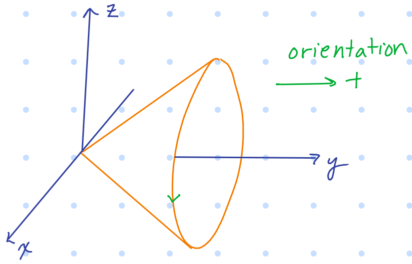
$$5 \cdot \pi 4^2$$

$$\boxed{80\pi}$$

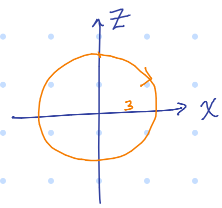
ex 3) Use Stokes' theorem to evaluate $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$ for

$\vec{F} = \langle x^2y^3z,\ \sin(xyz),\ xyz \rangle$

S = the part of $y^2 = x^2 + z^2$ between $y=0$ and $y=3$ oriented in the direction of the positive y-axis.



Boundary of S is the rim of the cone, ie the circle with this orientation below



Stokes': $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$

Parametrize C: $\vec{r}(t) = \langle 3\sin t,\ 3,\ 3\cos t \rangle \quad 0 \le t \le 2\pi$

Compute $\vec{r}\,'(t)$: $r'(t) = \langle 3\cos t,\ 0,\ -3\sin t \rangle$

Compute $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$: $\langle \sin^2 t \cdot 3^6 \cdot \cos t,\ \sin(27\sin t\cos t),\ 27\sin t\cos t \rangle \cdot \langle 3\cos t,\ 0,\ -3\sin t \rangle$

$3^7 \sin^2 t\cos^2 t - 81\sin^2 t\cos t$

Write integral: $\int_0^{2\pi} 3^7 \sin^2 t\cos^2 t - 81\sin^2 t\cos t\ dt$

$\boxed{\frac{2187}{4}\pi}$

Name: ____

1. Determine whether each statement is TRUE or FALSE

   (a) If $\vec{F}$ is a vector field, then div $\vec{F}$ is a vector field.

   (b) If $\vec{F}$ is a vector field and curl $\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

   (c) If S is a sphere and $\vec{F}$ is a constant vector field, then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

   (d) The line integral $\int_C \vec{F} \cdot d\vec{r}$ is a scalar.

   (e) If C_1 and C_2 are oriented curves and the length of C_1 is greater than the length of C_2, then $\int_{C_1} \vec{F} \cdot d\vec{r} > \int_{C_2} \vec{F} \cdot d\vec{r}$.

   (f) If $\vec{F} = \nabla f$, then $\vec{F}$ is path-independent.

   (g) The value of a flux integral is a scalar.

   (h) The flux of the vector field $\vec{F} = \langle 1, 0, 0 \rangle$ through the plane $x = 0$, $0 \le y \le 1$, $0 \le z \le 1$, oriented in the positive x-direction is zero.

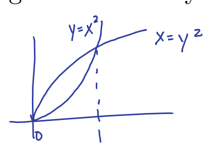
2. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

   (a) $\vec{F}(x, y, z) = \langle y^2 \cos z, 2xy \cos z, -xy^2 \rangle$
   $C : \vec{r}(t) = \langle t^2, \sin t, t \rangle$, $0 \le t \le \pi$

   Kinda ugly, maybe typo?

   (b) $\vec{F}(x, y) = \langle y + e^{\sqrt{x}}, 2x + \cos y^2 \rangle$
   C is the boundary of the region enclosed by parabolas $y = x^2$ and $x = y^2$

   Is $\vec{F}$ conservative?
   $Q_x - P_y \neq 0$, so no
   $\ddot{\frown}$
   Must compute with brute force or Green's theorem. Let's use Green's.



   $\int_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$ (Green's thm)
   $= \int_0^1 \int_{x^2}^{\sqrt{x}} 2 - 1 \, dy\, dx$
   $= \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx$

   (c) $\vec{F}(x, y, z) = \langle e^y, xe^y, (z+1)e^z \rangle$
   $C : \vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$

   Is $\vec{F}$ conservative?
   curl $\vec{F} = \vec{0}$, so yes!!
   Use Fundamental Thm of Line Integrals!

   Find potential function f
   $\left.\begin{array}{l} \int e^y \, dx = e^y \cdot x + h(y,z) \\ \int xe^y \, dy = xe^y + g(x,z) \\ \int (z+1)e^z \, dz = ze^z + p(x,y) \end{array}\right\} f(x,y,z) = xe^y + ze^z$

   Use FTLI
   $\int_C \nabla f \cdot d\vec{r} = f(\text{end pt}) - f(\text{start pt})$
   $= f(1,1,1) - f(0,0,0)$
   $= 1 \cdot e^1 + 1 \cdot e^1 - 0$
   $= \boxed{2e}$

   (d) $\vec{F}(x, y) = \langle e^x + x^2 y, e^y - xy^2 \rangle$
   C is the circle $x^2 + y^2 = 25$ oriented clockwise.

   Is $\vec{F}$ conservative?
   $Q_x - P_y \neq 0$, so no
   $\ddot{\frown}$
   Use brute force or Green's theorem. Let's use Green's

   $\int_{-C} F \cdot dr = -\iint_D Q_x - P_y \, dA$
   $= -\iint_D -y^2 - x^2 \, dA$
   $= \int_0^5 \int_0^{2\pi} r^2 \cdot r \, d\theta \, dr$

3. Is $\vec{F}(x, y, z) = \langle e^z, 1, xe^z \rangle$ a conservative vector field? If so, find f such that $\vec{F} = \nabla f$.

No special theorems available for integral of scalar function
Use brute force.

4. Evaluate $\iint_S x^2+y^2\,dS$ where S is the surface with vector equation $\vec{r}(u,v)=\langle 2uv, u^2-v^2, u^2+v^2\rangle$, $u^2+v^2\le 1$.

$\iint_S f\,dS = \iint_{u,v} f(\vec{r}(u,v))\,|\vec{r}_u\times\vec{r}_v|\,dA$

Compute $\vec{r}_u\times\vec{r}_v$

$\vec{r}_u = \langle 2v, 2u, 2u\rangle$

$\vec{r}_v = \langle 2u, -2v, 2v\rangle$

$\vec{r}_u\times\vec{r}_v = \langle 4uv+4uv, -(4v^2-4u^2), -4v^2-4u^2\rangle$

$|\vec{r}_u\times\vec{r}_v| = 4\sqrt{2}(v^2+u^2)$

Write integral

$\iint \left(4u^2v^2+(u^2-v^2)^2\right)4\sqrt{2}(v^2+u^2)\,dA$

$4\sqrt{2}\iint (u^4+2u^2v^2+v^4)\,(v^2+u^2)\,dA$

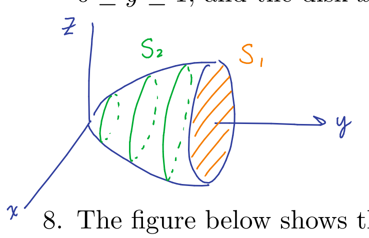
$4\sqrt{2}\iint (u^2+v^2)^3\,dA$

$4\sqrt{2}\int_0^{2\pi}\int_0^1 (r^2)^3\cdot r\,dr\,d\theta$

5. Evaluate $\iint_S z+x^2y\,dS$ where S is the part of the cylinder $x^2+y^2=1$ that lies between the planes $z=0$ and $z=3$ in the first octant.

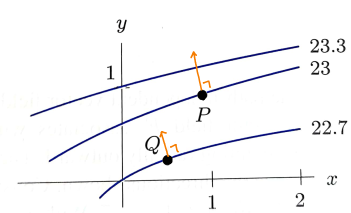
6. Evaluate the flux $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z)=\langle xze^y, -xze^y, z\rangle$ where S is the part of the plane $x+y+z=1$ in the first octant with downward orientation.

7. Evaluate the flux $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z)=\langle 0, y, -z\rangle$ where S is the part of the paraboloid $y=x^2+z^2$, $0\le y\le 1$, and the disk $x^2+z^2\le 1$, $y=1$.



Is $\vec{F}$ conservative?
curl $\vec{F}=\vec{0}$ so yes
Since S is a closed surface and $\vec{F}$ is conservative,
$\iint_S \vec{F}\cdot d\vec{S} = 0$

8. The figure below shows the level curves of $f(x,y)$.



(a) Sketch ∇f at P.

(b) Is the vector ∇f at P longer than, shorter than, or the same length as ∇f at Q?

(c) If C is a curve from P to Q, evaluate $\int_C \nabla f\cdot d\vec{r}$.

By Fundamental Thm of Line integrals,
$\int_C \nabla f\cdot d\vec{r} = f(\text{end pt}) - f(\text{start pt}) = 22.7 - 23 = -.3$

Note: For more great review problems, see exercises at the end of Chapter 13, questions 11-28

Answers
(1) F F T T F T T F
(2) (a) 0 (b) 1/3 (c) 2e (d) $625\pi/2$
(3) $f = xe^z + y + C$
(4) $\sqrt{2}\pi$
(5) 12
(6) $-1/6$
(7) 0
(8) (a) arrow from P pointing up and slightly left, perpendicular to the level curve (b) vector at P is longer (c) -0.3

Compute directional derivative of $f(x,y) = x^2y + e^x$ at $(0,0)$ in the direction $\langle 4,-1\rangle$

$\nabla f = \langle 2xy + e^x, x^2\rangle$ $|v| = \sqrt{4^2+1^2} = \sqrt{17}$

$\nabla f(0,0) = \langle 1, 0\rangle$

$$D_{\vec{u}} f(0,0) = \langle 1,0\rangle \cdot \langle 4,-1\rangle \left(\frac{1}{\sqrt{17}}\right) = 4/\sqrt{17}$$

Monday November 30

Reminders

- Quiz 7, 8 corrections due tonight before midnight
- Fill out FCQs
- Fill out feedback form on Google Forms
- WebAssign 13.7
- HW 14 Section 13.7

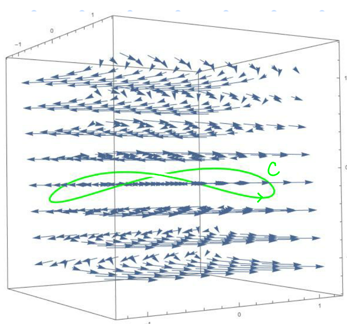
13.7 Stokes' Theorem (cont)

Recap

Stokes' theorem: For oriented surface S with positively oriented boundary C,

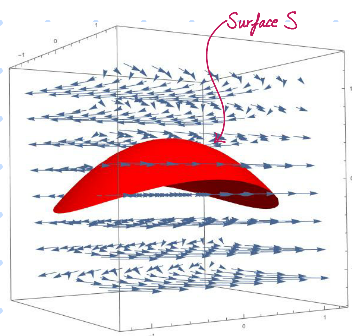
$$\int_{C=\partial S} \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

Stokes' says these integrals are equal



Integral of $\vec{F}$ along the curve C

$=$



Integral of curl $\vec{F}$ over surface S

ex 4) Let $\vec{F}=\nabla f$ for $f(x,y,z)=y^2\sin(xz)$. Evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ when S is the part of $z=xy$ on $0\le x\le 1$, $0\le y\le 1$

$\vec{F}=\nabla f$, which means $\vec{F}$ is conservative.

So $\text{curl}\,\vec{F}=\vec{0}$ and $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \vec{0}\cdot d\vec{S} = \boxed{0}$

* Popular exam question

ex 5) Evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ for $\vec{F} = \langle e^{xy}\cos z,\ x^2 z,\ xy \rangle$ and S is $x = \sqrt{1-y^2-z^2}$ oriented in the direction of positive x.

Options for computation

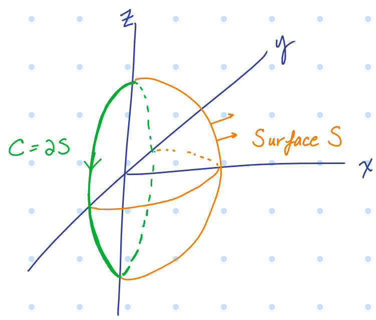
(a) Brute force $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{u,v} \text{curl}\,\vec{F}(\vec{r}(u,v))\cdot(r_u \times r_v)\, dA$

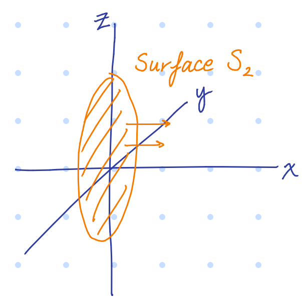
(b) Stokes' theorem via line integral $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}$

(c) Stokes' theorem via a different surface $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}$

Is $\vec{F}$ conservative? No ☹

Sketch S and $\partial S = C$





Choose option (c) with S_2 as the flat disk given by $\vec{r}(u,v) = \langle 0, y, z \rangle$ $\quad y^2 + z^2 \le 1$

Write parametrized form $\iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{u,v} \text{curl}\,\vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times r_v)\, dA$

Find $\vec{r}_u \times \vec{r}_v$

By inspection, let our unit normal be $\langle 1,0,0 \rangle$

Since $\vec{r}$ uses a rectangular parametrization, $|\vec{r}_u \times \vec{r}_v| = 1$

So $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

$$\langle 1,0,0 \rangle = \frac{\vec{r}_u \times \vec{r}_v}{1}$$

$$\vec{r}_u \times \vec{r}_v = \langle 1,0,0 \rangle$$

Compute curl $\vec{F}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xy}\cos z & x^2 z & xy \end{vmatrix}$$

$$= \Big\langle x - x^2,\ \underbrace{-(y + e^{xy}\sin z),\ 2xz - xe^{xy}\cos z}_{\text{irrelevant!}} \Big\rangle$$

Compute $\text{curl}\,\vec{F}(\vec{r}(u,v))\cdot \vec{r}_u \times \vec{r}_v$ $\quad \langle 1 - 1^2,\ \text{something},\ \text{something} \rangle \cdot \langle 1, 0, 0 \rangle = 0$

Write integral $\displaystyle\iint_{y^2+z^2\le 1} 0\, dA = \boxed{0}$

List of ways to use Stokes' theorem $\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$

1.) Find $\oint_C \vec{F} \cdot d\vec{r}$ by inventing a surface S with boundary C and proper orientation

Compute $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$ instead

2) Find $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$ by finding its boundary C with correct orientation

Compute $\oint_C \vec{F} \cdot d\vec{r}$ instead

3) Find $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$ by inventing a new surface S_2 with same boundary and orientation as S.

Compute $\iint_{S_2} \text{curl}\, \vec{F} \cdot d\vec{S}$ instead.

ex 6) Evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ where $\vec{F}=\langle z^2, -3xy, x^3y^3\rangle$ and S is the part of $z=5-x^2-y^2$ satisfying $z\geq 1$ with positive orientation in the positive z-direction

Options for computation

1.) Brute force

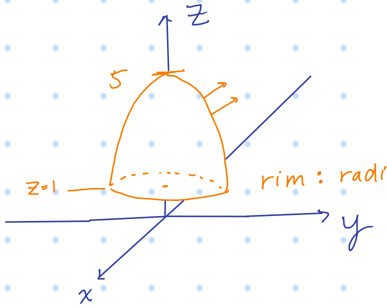
2.) Swap to line integral along boundary

[This is a valid choice. See next page to see the result if you picked this method]

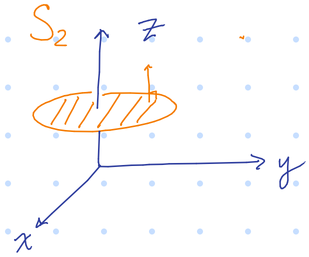
3.) Swap to a different surface

[I'll do this one below]

Sketch S



New surface S_2



Parametrize S_2 $\quad \vec{r}(u,v)=\langle u, v, 1\rangle, \quad u^2+v^2\leq 4$

Calculate $\vec{r}_u\times\vec{r}_v$ $\quad$ By inspection, we see $\vec{r}_u\times\vec{r}_v=\langle 0, 0, 1\rangle$

Compute curl $\vec{F}$

$$\text{curl}\,\vec{F}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & -3xy & x^3y^3 \end{vmatrix} = \langle \underbrace{??, ??}, -3y\rangle$$

These will disappear when we dot with $\langle 0,0,1\rangle$ so they don't matter

Write integral

$$\iint_{u^2+v^2\leq 4} \langle ??, ??, -3v\rangle\cdot\langle 0,0,1\rangle\, dA$$

$$\iint -3v\, dA$$

$$\int_0^{2\pi}\int_0^2 -3r\sin\theta\cdot r\, dr\, d\theta = \boxed{0}$$

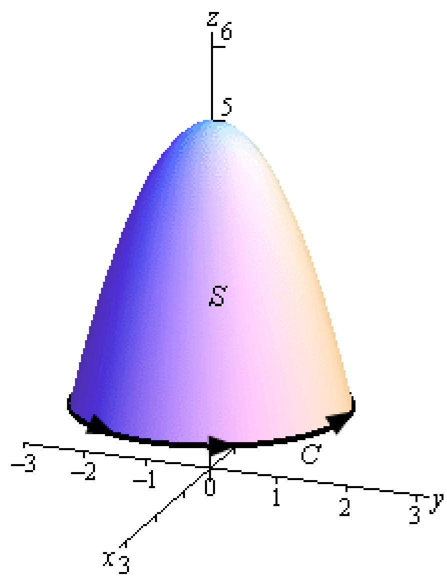
Here are some excellent worked examples from Paul's Online Notes.
Original document found at
https://tutorial.math.lamar.edu/pdf%5CCalcIII%5CCalcIII_StokesThm.pdf

Let's take a look at a couple of examples.

Example 1 Use Stokes' Theorem to evaluate $\iint\limits_S \operatorname{curl}\vec{F}\bullet d\vec{S}$ where $\vec{F} = z^2\,\vec{i} - 3xy\,\vec{j} + x^3y^3\,\vec{k}$ and *S* is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that *S* is oriented upwards.

Solution
Let's start this off with a sketch of the surface.

Same problem as previous example but different method



In this case the boundary curve *C* will be where the surface intersects the plane $z = 1$ and so will be the curve

$$\begin{aligned} 1 &= 5 - x^2 - y^2 \\ x^2 + y^2 &= 4 \qquad\qquad \text{at } z = 1 \end{aligned}$$

So, the boundary curve will be the circle of radius 2 that is in the plane $z = 1$. The parameterization of this curve is,

$$\vec{r}\left(t\right) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + \vec{k}, \quad 0 \le t \le 2\pi$$

The first two components give the circle and the third component makes sure that it is in the plane $z = 1$.

Using Stokes' Theorem we can write the surface integral as the following line integral.

$$\iint\limits_S \operatorname{curl}\vec{F}\bullet d\vec{S} = \int\limits_C \vec{F}\bullet d\,\vec{r} = \int_0^{2\pi} \vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}\,'\left(t\right)dt$$

So, it looks like we need a couple of quantities before we do this integral. Let's first get the vector field evaluated on the curve. Remember that this is simply plugging the components of the parameterization into the vector field.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right) &= \left(1\right)^2\,\vec{i} - 3\left(2\cos t\right)\left(2\sin t\right)\vec{j} + \left(2\cos t\right)^3\left(2\sin t\right)^3\vec{k}\\ &= \vec{i} - 12\cos t\sin t\,\vec{j} + 64\cos^3 t\sin^3 t\,\vec{k}\end{aligned}$$

Next, we need the derivative of the parameterization and the dot product of this and the vector field.

$$\vec{r}'\left(t\right) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$$

$$\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}'\left(t\right) = -2\sin t - 24\sin t\cos^2 t$$

We can now do the integral.

$$\begin{aligned}\iint\limits_S \operatorname{curl}\vec{F}\bullet d\vec{S} &= \int_0^{2\pi} -2\sin t - 24\sin t\cos^2 t\,dt\\ &= \left.\left(2\cos t + 8\cos^3 t\right)\right|_0^{2\pi}\\ &= 0\end{aligned}$$

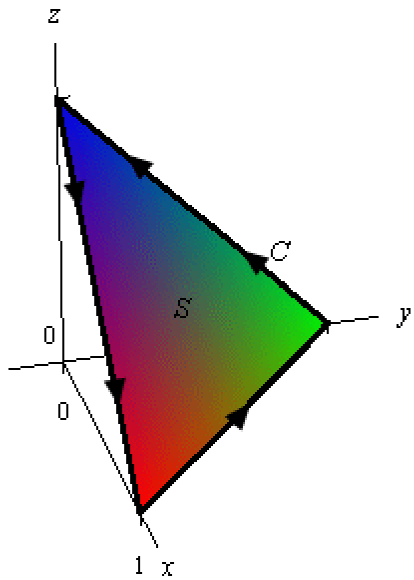
Example 2 Use Stokes' Theorem to evaluate $\int\limits_C \vec{F}\bullet d\,\vec{r}$ where $\vec{F} = z^2\,\vec{i} + y^2\,\vec{j} + x\,\vec{k}$ and *C* is the triangle with vertices $\left(1,0,0\right)$, $\left(0,1,0\right)$ and $\left(0,0,1\right)$ with counter-clockwise rotation.

Solution

We are going to need the curl of the vector field eventually so let's get that out of the way first.

$$\operatorname{curl}\vec{F} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ z^2 & y^2 & x\end{vmatrix} = 2z\,\vec{j} - \vec{j} = \left(2z-1\right)\vec{j}$$

Now, all we have is the boundary curve for the surface that we'll need to use in the surface integral. However, as noted above all we need is any surface that has this as its boundary curve. So, let's use the following plane with upwards orientation for the surface.



Since the plane is oriented upwards this induces the positive direction on *C* as shown. The equation of this plane is,

$$x + y + z = 1 \qquad \Rightarrow \qquad z = g\left(x, y\right) = 1 - x - y$$

Now, let's use Stokes' Theorem and get the surface integral set up.

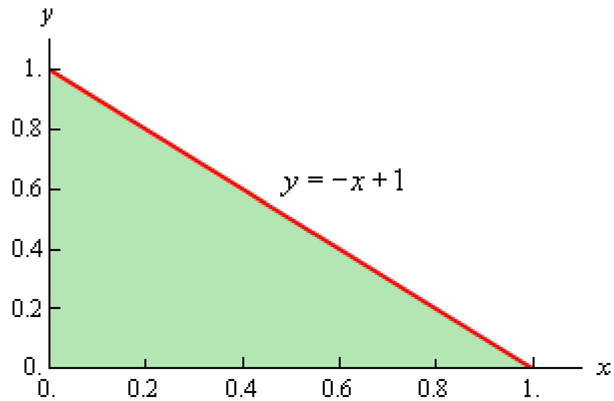
$$\begin{aligned}\int_C \vec{F} \bullet d\,\vec{r} &= \iint_S \operatorname{curl} \vec{F} \bullet d\vec{S} \\ &= \iint_S \left(2z - 1\right) \vec{j} \bullet d\vec{S} \\ &= \iint_D \left(2z - 1\right) \vec{j} \bullet \frac{\nabla f}{\left\| \nabla f \right\|} \left\| \nabla f \right\| dA\end{aligned}$$

Okay, we now need to find a couple of quantities. First let's get the gradient. Recall that this comes from the function of the surface.

$$f\left(x, y, z\right) = z - g\left(x, y\right) = z - 1 + x + y$$
$$\nabla f = \vec{i} + \vec{j} + \vec{k}$$

Note as well that this also points upwards and so we have the correct direction.

Now, *D* is the region in the *xy*-plane shown below,



We get the equation of the line by plugging in $z = 0$ into the equation of the plane. So based on this the ranges that define *D* are,

$$0 \le x \le 1 \qquad 0 \le y \le -x + 1$$

The integral is then,

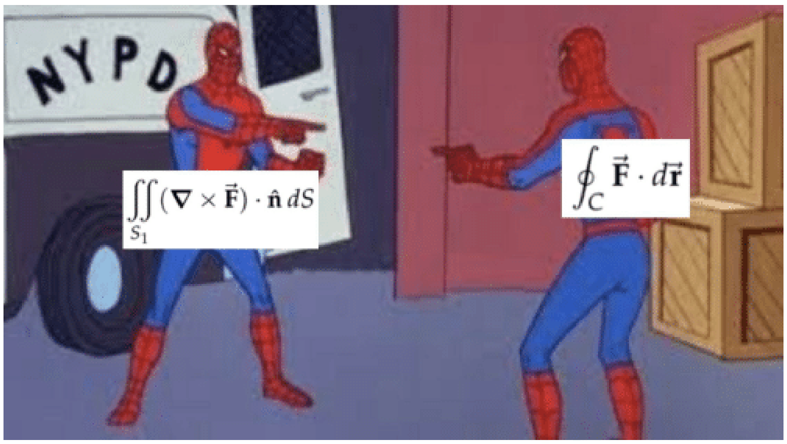
$$\begin{aligned}\int_C \vec{F} \bullet d\,\vec{r} &= \iint_D (2z - 1)\,\vec{j} \bullet \left(\vec{i} + \vec{j} + \vec{k}\right) dA \\ &= \int_0^1 \int_0^{-x+1} 2(1 - x - y) - 1\,dy\,dx\end{aligned}$$

Don't forget to plug in for *z* since we are doing the surface integral on the plane. Finishing this out gives,

$$\begin{aligned}\int_C \vec{F} \bullet d\,\vec{r} &= \int_0^1 \int_0^{-x+1} 1 - 2x - 2y\,dy\,dx \\ &= \int_0^1 \left(y - 2xy - y^2\right)\Big|_0^{-x+1} dx \\ &= \int_0^1 x^2 - x\,dx \\ &= \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_0^1 \\ &= -\frac{1}{6}\end{aligned}$$

In both of these examples we were able to take an integral that would have been somewhat unpleasant to deal with and by the use of Stokes' Theorem we were able to convert it into an integral that wasn't too bad.

Fill out FCQs and feedback forms, please



Good old Stokes Theorem

Tuesday December 1

Reminders

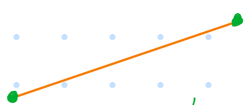
- Nothing much! Calculate grades? Make a study plan?

13.8 Divergence theorem

Refresher on boundaries + orientation

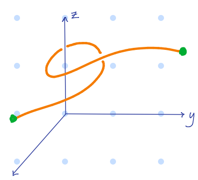
What is the boundary of each shape?

(a) line segment



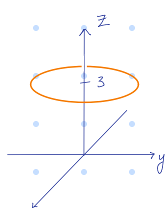
two points

(b) space curve that's not closed



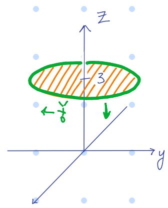
two points

(c) circle $x^2+y^2=1,\ z=3$



no boundary

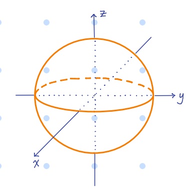
(d) disk $x^2+y^2\le 1,\ z=3$



circle on the rim

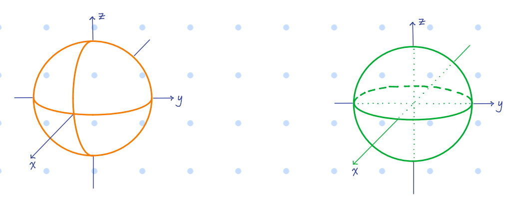
if disk is oriented downward,
boundary is oriented clockwise
when viewed from above

(e) sphere $x^2+y^2+z^2=1$



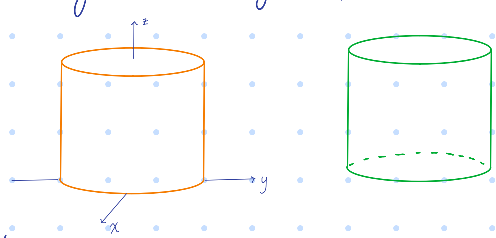
no boundary

(f) solid ball $x^2+y^2+z^2\le 1$



sphere

(g) solid cylinder $x^2+y^2\le 1,\ 0\le z\le 3$



two disks and
a tube

(h) the 4D ball $x^2+y^2+z^2+w^2\le 1$

??

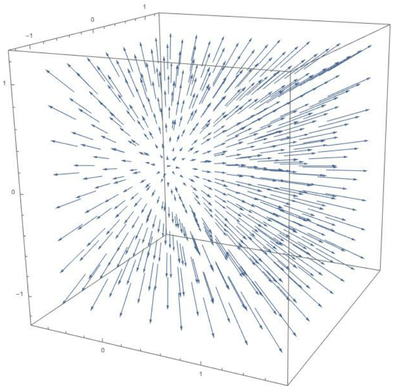
3 dimensional

2 dimensional

Divergence Theorem: Let E be a simple solid region and let the surface S be the boundary of E with positive outward orientation. Let $\vec{F}$ be a vector field with continuous partial derivatives on an open region containing E. Then

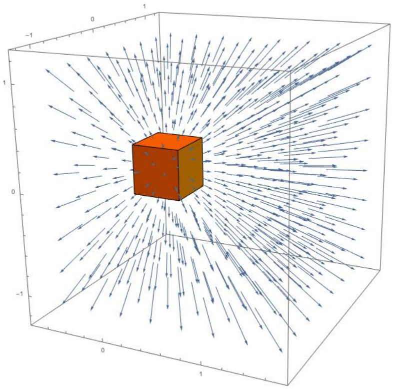
$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

$$dV \rightarrow \begin{cases} dz\,dy\,dx \\ r\,dz\,dr\,d\theta \\ \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \end{cases}$$



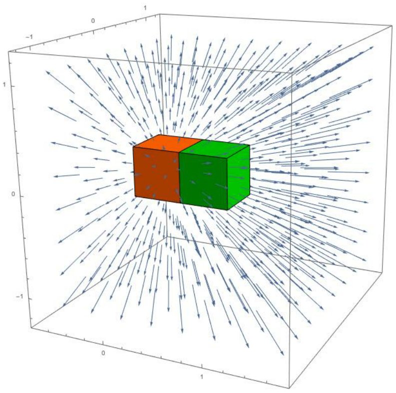
This vector field is $\vec{F} = \langle x, y, z \rangle$

The divergence of $\vec{F}$ is $\operatorname{div} F = 3$



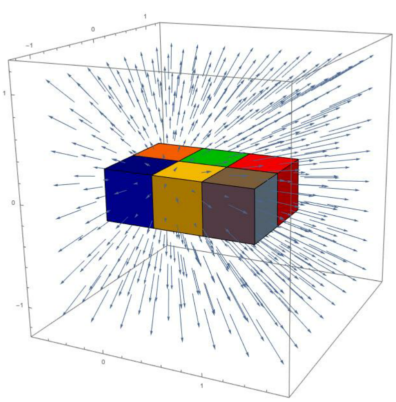
The outward flux of $\vec{F}$ across the surface given by the boundary of the cube is positive

The integral of $\operatorname{div}\vec{F}$ over the solid cube is $\iiint_{\text{cube}} 3 \, dV$
The value here is positive.



The outward flux of $\vec{F}$ across the surface given by the boundary of the joined double-cube is equal to (flux out of orange cube) + (flux out of green cube). The total flux is positive.
Notice that the joined sides cancel!

The integral of $\operatorname{div}\vec{F}$ over the joined double-cube is $\iiint_{\substack{\text{double} \\ \text{cube}}} 3 \, dV$. The value here is positive.



The outward flux of $\vec{F}$ across the surface given by the boundary of the joined mega-cube is equal to the sum of the fluxes out of each small cube. The total flux is positive.
Notice that the joined sides cancel!

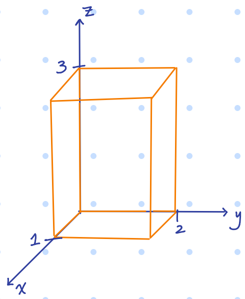
The integral of $\operatorname{div}\vec{F}$ over the joined mega-cube is $\iiint_{\substack{\text{mega} \\ \text{cube}}} 3 \, dV$. The value here is positive.

ex 1) Calculate the flux of $\vec{F}$ across S

$\vec{F} = \langle x^2yz,\ xy^2z,\ xyz^2 \rangle$

S is the surface of the box given by $0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3$.

Sketch S



Options for computing

1.) Brute force via $\iint_S \vec{F} \cdot d\vec{S} = \iint_{u,v} \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dA$

But S has six pieces, so that's 6 integrals

2.) If $\vec{F}$ happens to be curl of some other VF $\vec{G}$, then

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \text{curl}\,\vec{G} \cdot d\vec{S} = \int_{\partial S} \vec{G} \cdot d\vec{r}$ (by Stokes' thm)

No clue if $\vec{F} = \text{curl}\,\vec{G}$. Can't do this method

3) Divergence thm says $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\, dV$

E is a really nice shape!

Do this one.

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q + \frac{\partial}{\partial z} R$$

$$= 2xyz + 2xyz + 2xyz$$

$$= 6xyz$$

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iiint_E 6xyz \, dV$$

$$= \int_0^1 \int_0^2 \int_0^3 6xyz \, dz\, dy\, dx$$

$$= \boxed{27}$$

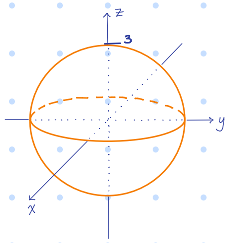
ex 2) Calculate the flux of $\vec{F}$ across S

$\vec{F} = \langle x^3+y^3,\ y^3+z^3,\ z^3+x^3 \rangle$

S is the sphere centered at the origin with radius 3

* Common exam question

Sketch S



Options

1.) Brute force: $\iint_S \vec{F}\cdot d\vec{S} = \iint_{u,v} \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dA$

Seems clunky

2.) Is $\vec{F} = \text{curl}\, \vec{G}$ so that Stokes' theorem applies? No clue!

3.) Divergence thm seems promising.

$$\text{div}\, \vec{F} = \frac{\partial}{\partial x}P + \frac{\partial}{\partial x}Q + \frac{\partial}{\partial z}R$$
$$= 3x^2 + 3y^2 + 3z^2$$

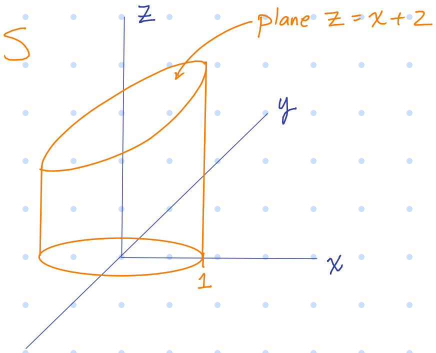
$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S}$$
$$= 3\iiint_E x^2+y^2+z^2 \, dV$$
$$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$
$$= 3\cdot 2\pi \cdot \frac{1}{5}(3^5)\cdot 2$$
$$= \boxed{\frac{2916\pi}{5}}$$

ex 3) Calculate the flux of $\vec{F}$ across S

$\vec{F} = \langle x^4, -x^3z^2, 4xy^2z \rangle$

S is the surface of the solid bounded by $x^2+y^2=1$, $z=x+2$, $z=0$

Sketch S



$\operatorname{div}\vec{F} = 4x^3 + 0 + 4xy^2$

$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S}$$

$$= \iiint_E \operatorname{div}\vec{F}\, dV$$

$$= 4\iiint_E x(x^2+y^2)\, dV$$

$$= 4\int_0^{2\pi}\int_0^1\int_0^{r\cos\theta+2} r\cos\theta \cdot r^2 \cdot r\, dz\, dr\, d\theta$$

$$= \boxed{\frac{2\pi}{3}}$$

Wednesday December 2

Reminders

- Compile HW 14 for André
- WebAssign 13.8

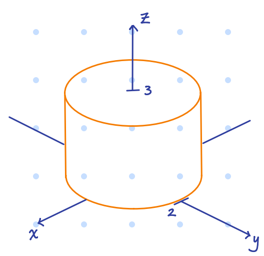
13.8 Divergence theorem (cont)

Recap

Divergence theorem: $\iint_{S=\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$

ex 4) Find the flux of $\vec{F} = \langle \cos(z), \sin(z), x^2+y^2 \rangle$ across the boundary of the cylinder $x^2+y^2 \le 4$, $0 \le z \le 3$

Sketch S



Options for computing

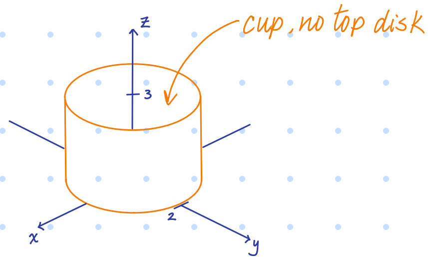
(a) Brute force - this would require 3 separate flux integrals
(b) Is $\vec{F}$ the curl of some $\vec{G}$ so that we can use Stokes'? Dunno!
(c) E is a nice shape. Divergence theorem looks good.

$$\begin{aligned} \operatorname{div} \vec{F} &= \tfrac{\partial}{\partial x} P + \tfrac{\partial}{\partial x} Q + \tfrac{\partial}{\partial z} R \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Flux} &= \iint_S \vec{F} \cdot d\vec{r} \\ &= \iiint_E 0 \, dV \\ &= 0 \end{aligned}$$

ex 4) Find the flux of $\vec{F} = \langle \cos(z), \sin(z), x^2+y^2 \rangle$ across the cylindrical cup given by the tube $x^2+y^2=4$, $0 \le z \le 3$ and the disk $x^2+y^2 \le 4$, $z=0$.

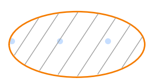
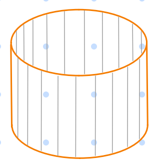
Sketch surface



The divergence theorem uses a solid E and its entire boundary, but our surface is only a piece of the boundary!

Solution: split up the boundary.

If E is the solid cylinder $x^2+y^2 \le 4$, $0 \le z \le 3$ then its boundary ∂E is made of a tube and two disks

$\partial E =$  and 

S_1 = top disk

S_2 = tube and bottom disk

[This is the surface we care about]

Then the divergence theorem says

$$\iiint_E \text{div}\, \vec{F} = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$

$$\iiint_E \text{div}\, \vec{F} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \boxed{\iint_{S_2} \vec{F} \cdot d\vec{S}}$$

we want this.

$$\underset{\text{(from previous example)}}{0} = \underbrace{\iint_{S_1} \vec{F} \cdot d\vec{S}_1}_{\text{Let's actually compute this.}} + \iint_{S_2} \vec{F} \cdot d\vec{S}_2$$

$\underline{\iint_{S_1} \vec{F} \cdot d\vec{S}_1}$

S_1 : $\vec{r}(u,v) = \langle u, v, 3 \rangle \quad u^2+v^2 \le 2$

$\vec{r}_u \times \vec{r}_v = \langle 0, 0, 1 \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle \cos 3, \sin 3, u^2+v^2 \rangle$

$\vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v = u^2+v^2$

$$\iint_{S_1} \vec{F} \cdot d\vec{S}_1 = \iint_{u,v} u^2+v^2 \, dA$$

$$= \int_0^{2\pi} \int_0^2 r^2 \cdot r \, dr \, d\theta$$

$$= 8\pi$$

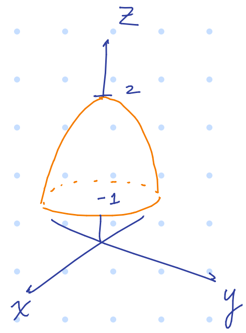
$$0 = 8\pi + \iint_{S_2} \vec{F} \cdot d\vec{S}$$

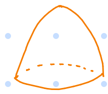
$$\boxed{\iint_{S_2} \vec{F} \cdot d\vec{S} = -8\pi}$$

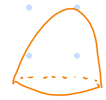
* Popular exam question

ex 5) Let $\vec{F} = \langle z\tan^{-1}(y^2),\ z^3\ln(x^2+1),\ z \rangle$. Find the flux of $\vec{F}$ across the paraboloid $x^2+y^2+z=2$ that lies above $z=1$ and is oriented upward.

Sketch



$E =$ solid 

$\partial E =$  and 

$S_1 =$ bell $\qquad$ $S_2 =$ disk

$$\underbrace{\iint_{S_1} \vec{F}\cdot d\vec{S}}_{\text{goal}} + \underbrace{\iint_{S_2} \vec{F}\cdot d\vec{S}}_{\text{compute}} = \underbrace{\iiint_E \operatorname{div}\vec{F}\, dV}_{\text{compute}}$$

$$\iint_{S_2} \vec{F}\cdot d\vec{S}$$

S_2: $\vec{r} = \langle u, v, 1 \rangle \qquad u^2+v^2 \le 1$

$\vec{r}_u \times \vec{r}_v = \langle 0, 0, -1 \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle \tan^{-1}(v^2), \ln(u^2+1), 1 \rangle$

$\vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v = -1$

$$\iint_{u,v} -1 \, dA$$

$$\int_0^{2\pi}\int_0^1 -r \, dr\, d\theta$$

$$-\pi$$

$$\iiint_E \operatorname{div}\vec{F}\, dV$$

$\operatorname{div}\vec{F} = 0 + 0 + 1$

$$\iiint 1 \, dV$$

$$\int_0^{2\pi}\int_0^1\int_0^{2-r^2} r \, dz\, dr\, d\theta$$

$$\pi/2$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} - \pi = \pi/2$$

$$\boxed{\iint_{S_1} \vec{F}\cdot d\vec{S} = \frac{3\pi}{2}}$$

The Real Stokes' theorem

Here's what the Wikipedia article on Stokes' theorem looks like:

Wikipedia Search Wikipedia

Stokes' theorem

Language Download PDF Watch Edit

This article is about the generalized theorem. For the classical theorem, see Kelvin–Stokes theorem. For the equation governing viscous drag in fluids, see Stokes' law.

In vector calculus and differential geometry, **Stokes' theorem** (sometimes spelled **Stokes's theorem**), also called the **generalized Stokes theorem** or the **Stokes–Cartan theorem**,[1] is a statement about the integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus. Stokes' theorem says that the integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative $d\omega$ over the whole of Ω, i.e.,

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega\,.$$

??!

Stokes' theorem was formulated in its modern form by Élie Cartan in 1945,[2] following earlier work on the generalization of the theorems of vector calculus by Vito Volterra, Édouard Goursat, and Henri Poincaré.[3][4]

Let's compare it to the Stokes' theorem we know

Our version of Stokes' theorem:

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

the integrand and some kind of derivative

A shape and its boundary

Wikipedia version of Stokes' theorem:

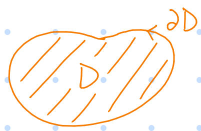
$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

the integrand and some kind of derivative

A shape and its boundary

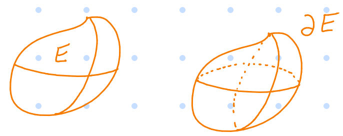
But wait... there's more!

Green's theorem:



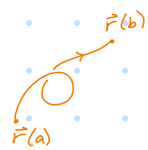
$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$$

Divergence theorem:



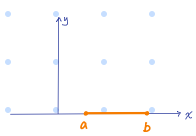
$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \, dV$$

Fundamental theorem of line integrals:



$$f \Big|_{\vec{r}(a)}^{\vec{r}(b)} = f(\vec{r}(b)) - f(\vec{r}(a)) = \int_C \nabla f \cdot d\vec{r}$$

Fundamental theorem of calculus:



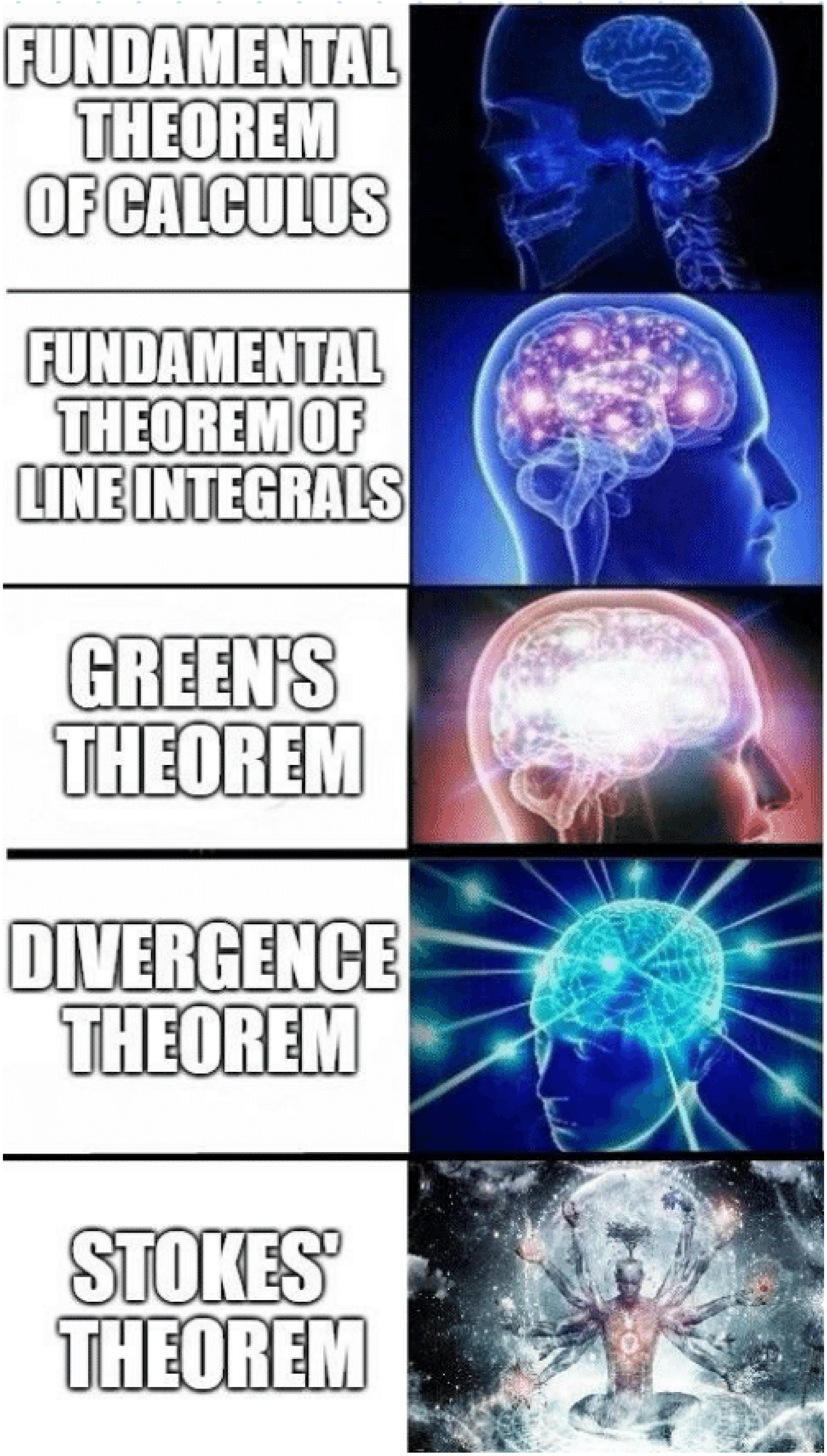
$$F(x) \Big|_a^b = F(b) - F(a) = \int_a^b F'(x) \, dx$$

The same pattern holds for all of these theorems!!!

In the Wikipedia version of Stokes theorem, the "derivative" is some higher notion called the exterior derivative that becomes div, curl, grad, or $\frac{d}{dx}$ in low dimensions.

$$\int_{\partial\Omega} \omega = \int_{\Omega} \underbrace{d\omega}_{\text{exterior derivative}}$$

So all of these theorems are just special cases of the one true Stokes' theorem!



Fundamental meme of calculus

Friday December 3

Reminders

- All Week 14 WebAssign due Sunday before midnight
- Optional Quiz 9 review on WebAssign

Go to student.desmos.com and use code QDC AC8
Do the "FTC Matching" activity

This page left blank for you to try. Solutions attached 2 pages onward.

Part I - Summary of higher-dimensional versions of the Fundamental Theorem of Calculus
Fill in the blanks (assuming appropriate hypotheses are met for the integrands).

1. The theorem regarding the equation $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div}\mathbf{F}\, dV$ can be stated as follows:

   Given a surface integral of a vector field $\mathbf{F}$ over a surface S, if the surface S is ________, the surface integral is equal to a ________________ of _______ over the region bounded by the surface. (_____________ Theorem)

2. Given a line integral of a vector field $\mathbf{F}$ over a curve C, if $\mathbf{F}$ is ______________, then the value of the line integral is the difference between f evaluated at the start point and end point of the curve, where ____f = ____. (________________________)

3. Given a line integral of a vector field $\mathbf{F}$ along a curve C, if the curve C is ________, the line integral is equal to a __________________ of _________ over *any* orientable surface that has the curve C as its boundary. (____________________)

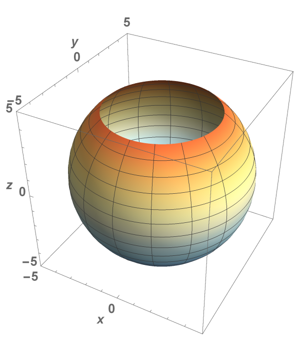
4. Given a line integral of a vector field $\mathbf{F} = \langle P, Q\rangle$ over a planar closed curve C (oriented counter-clockwise), the line integral is equal to a _________________ of ____________ over the planar region bounded by C. (____________________)

5. To evaluate $\iiint_E \nabla\cdot\mathbf{F}\, dV$, you can calculate $\iint_S$ ________, where S is ______________________________. (________________________)

6. To evaluate $\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$ (over an orientable surface S), you can calculate $\int_C$ ________, where C is ________________________________. (___________________)

Part II - Practice problems

1. The figure below shows a surface S, which is a sphere of radius 5 centered at the origin, with the top cut off, so the upper edge of the surface lies at $z = 4$. Use one of the theorems from Chapter 13 to evaluate $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = \langle y, -x, z\rangle$. **S is oriented outward.**



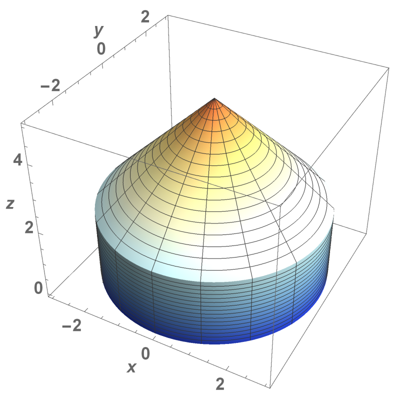
2. Consider the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the closed yurt-shaped surface shown below, and $\mathbf{F} = \langle 3x, 2y, z\rangle$. Notice that the surface comprises three separate pieces: the circular base, the cylinder walls, and the conical top. The cylinder has radius 3 and height 2, and the cone has radius 3 and height 3.

   (a) Discuss with your group the list of steps required to evaluate this surface integral directly.

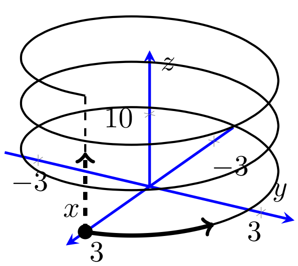
   (b) Use one of the theorems from Chapter 13 to set up a different type of integral with the same value as the given surface integral.

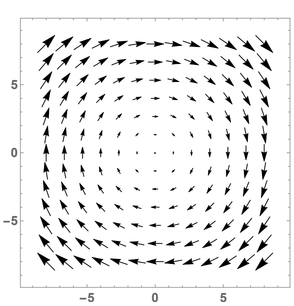
   (c) Interpret the new integral geometrically to find its value without evaluating it.

   S is oriented outward.



3. Consider the two integrals $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle y, -x, 0\rangle$, and where C_1 is shown below (solid), and C_2 is shown below (dashed). A top-view of the vector field $\mathbf{F}$ is also shown. Do the two line integrals give the same value, or not? Explain.





Part I - Summary of higher-dimensional versions of the Fundamental Theorem of Calculus
Fill in the blanks (assuming appropriate hypotheses are met for the integrands).

1. Given a surface integral of a vector field $\mathbf{F}$ over a surface S, if the surface S is closed, the surface integral is equal to a triple integral of $\text{div}\,\mathbf{F}$ over the region bounded by the surface. (Divergence Theorem)

2. Given a line integral of a vector field $\mathbf{F}$ over a curve C, if $\mathbf{F}$ is conservative, then the value of the line integral is the difference between f evaluated at the start point and end point of the curve, where $\nabla f = \mathbf{F}$. (FTC for line integrals)

3. Given a line integral of a vector field $\mathbf{F}$ along a curve C, if the curve C is closed, the line integral is equal to a surface integral of $\nabla \times \mathbf{F}$ over *any* orientable surface that has the curve C as its boundary. (Stokes' Theorem)

4. Given a line integral of a vector field $\mathbf{F} = \langle P, Q\rangle$ over a planar closed curve C (oriented counter-clockwise), the line integral is equal to a double integral of $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ over the planar region bounded by C. (Green's Theorem)

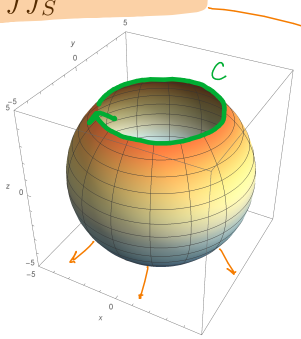
5. To evaluate $\displaystyle\iiint_E \nabla \cdot \mathbf{F}\, dV$, you can calculate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the boundary of the solid E. (Divergence Theorem)

6. To evaluate $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ (over an orientable surface S), you can calculate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the boundary of the surface S. (Stokes' Theorem)

Part II - Practice problems

Theorems: $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ || $\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ || $\iint_{\partial E} \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\,F\, dV$

1. The figure below shows a surface S, which is a sphere of radius 5 centered at the origin, with the top cut off, so the upper edge of the surface lies at $z = 4$. Use one of the theorems from Chapter 13 to evaluate $\displaystyle\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = \langle 2y, x, z\rangle$.



looks like Stokes' theorem (assume outward orientation b/c the problem doesn't specify)

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}$

$= \int_0^{2\pi} 2\cos^2 t - \sin^2 t\, dt$

$= \pi$

Parametrize ∂S: $\vec{r}(t) = \langle \sin t, \cos t, 4\rangle$, $0 \le t \le 2\pi$

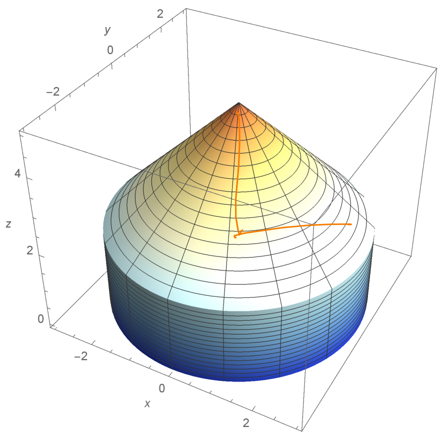
Compute $\vec{r}'(t)$: $\vec{r}'(t) = \langle \cos t, -\sin t, 0\rangle$

Compute $\vec{F}(\vec{r}(t))\cdot r'(t)$:

$\langle 2\cos t, \sin t, 4\rangle \cdot \langle \cos t, -\sin t, 0\rangle$

$2\cos^2 t - \sin^2 t$

2. Consider the surface integral $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the closed yurt-shaped surface shown below, and $\mathbf{F} = \langle 3x, 2y, z\rangle$. Notice that the surface comprises three separate pieces: the circular base, the cylinder walls, and the conical top. The cylinder has radius 3 and height 2, and the cone has radius 3 and height 3.
   (a) Discuss with your group the list of steps required to evaluate this surface integral directly.
   (b) Use one of the theorems from Chapter 13 to set up a different type of integral with the same value as the given surface integral.
   (c) Interpret the new integral geometrically to find its value without evaluating it.



The surface S is closed, so this feels like the Divergence theorem to me.

$$\iint_S \vec{F}\cdot d\vec{S} = \underbrace{\iint_{\text{top cone}} \vec{F}\cdot d\vec{S} + \iint_{\text{side tube}} \vec{F}\cdot d\vec{S} + \iint_{\text{bottom disk}} \vec{F}\cdot d\vec{S}}_{\text{so much work!}}$$

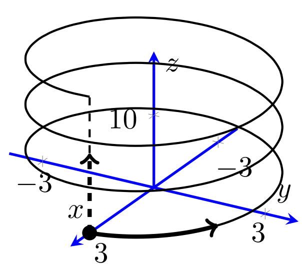
$$= \iiint_E \operatorname{div}\vec{F}\, dV$$

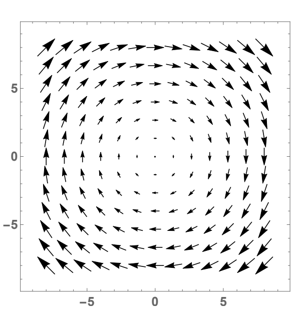
Compute $\operatorname{div}\vec{F}$: $3+2+1=6$

$$= \iiint 6\, dV$$

$$= 6\,(\text{volume of yurt})$$

3. Consider the two integrals $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ and $\displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle y, -x, 0\rangle$, and where C_1 is shown below (solid), and C_2 is shown below (dashed). A top-view of the vector field $\mathbf{F}$ is also shown. Do the two line integrals give the same value, or not? Explain.

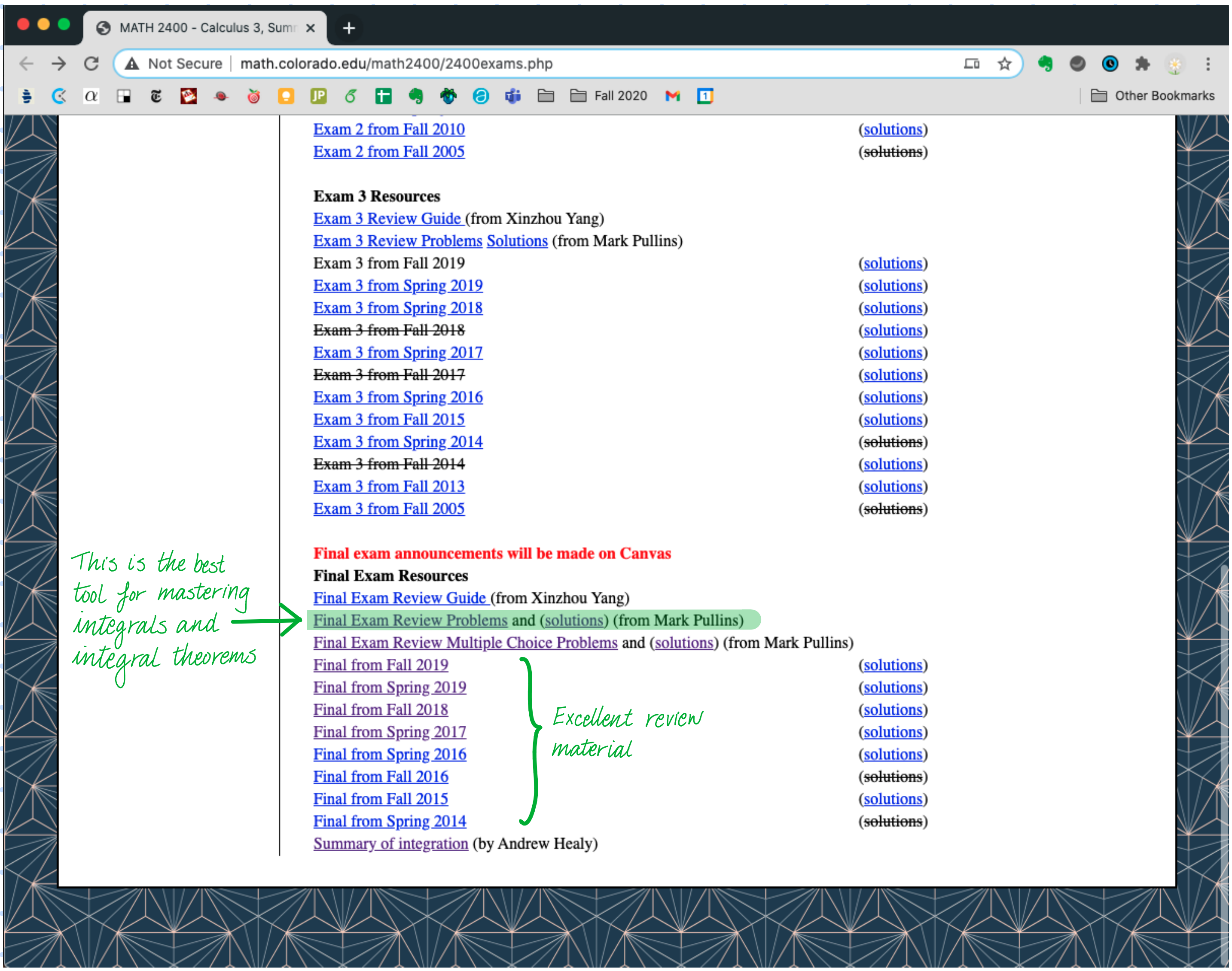




Monday December 7

Reminders

- Study for Quizzes 9 and 10
- [Important] Do Quizzes 9 and 10 on Wednesday between 7:30 and 10:00 AM


MATH 2400 - Calculus 3, Sumr
Not Secure | math.colorado.edu/math2400/2400exams.php
Fall 2020
Other Bookmarks

Exam 2 from Fall 2010	(solutions)
Exam 2 from Fall 2005	(~~solutions~~)

Exam 3 Resources

Exam 3 Review Guide (from Xinzhou Yang)

Exam 3 Review Problems Solutions (from Mark Pullins)

Exam 3 from Fall 2019	(solutions)
Exam 3 from Spring 2019	(solutions)
Exam 3 from Spring 2018	(solutions)
~~Exam 3 from Fall 2018~~	(solutions)
Exam 3 from Spring 2017	(solutions)
~~Exam 3 from Fall 2017~~	(solutions)
Exam 3 from Spring 2016	(solutions)
Exam 3 from Fall 2015	(solutions)
Exam 3 from Spring 2014	(~~solutions~~)
~~Exam 3 from Fall 2014~~	(solutions)
Exam 3 from Fall 2013	(solutions)
Exam 3 from Fall 2005	(~~solutions~~)

Final exam announcements will be made on Canvas

Final Exam Resources

Final Exam Review Guide (from Xinzhou Yang)

This is the best tool for mastering integrals and integral theorems

Final Exam Review Problems and (solutions) (from Mark Pullins)

Final Exam Review Multiple Choice Problems and (solutions) (from Mark Pullins)

Final from Fall 2019	(solutions)
Final from Spring 2019	(solutions)
Final from Fall 2018	(solutions)
Final from Spring 2017	(solutions)
Final from Spring 2016	(solutions)
Final from Fall 2016	(~~solutions~~)
Final from Fall 2015	(solutions)
Final from Spring 2014	(~~solutions~~)

Excellent review material

Summary of integration (by Andrew Healy)

Math 2400, Final Exam

December 18, 2019

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Shen Lu	8:00–8:50 AM
☐	Section 002	Euijin Hong	8:00–8:50 AM
☐	Section 003	Xingzhou Yang	8:00–8:50 AM
☐	Section 004	Mark Pullins	9:00–9:50 AM
☐	Section 005	Carly Matson	9:00–9:50 AM
☐	Section 006	Euijin Hong	9:00–9:50 AM
☐	Section 007	Xingzhou Yang	10:00–10:50 AM
☐	Section 008	Harrison Stalvey	10:00–10:50 AM
☐	Section 009	Mengxiao Sun	1:00–1:50 PM
☐	Section 010	Hanson Smith	1:00–1:50 PM
☐	Section 011	Carla Farsi	2:00–2:50 PM
☐	Section 012	Harrison Stalvey	2:00–2:50 PM
☐	Section 013	Carla Farsi	3:00–3:50 PM
☐	Section 014	Mengxiao Sun	4:00–4:50 PM
☐	Section 015	Trevor Jack	4:00–4:50 PM

- No calculators or cell phones or other electronic devices allowed at any time.
- You are allowed two $3'' \times 5''$ index cards written on both sides.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 150 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like ln(3)/2 as is.
- For cylindrical coordinates use (r, θ, z), and for spherical coordinates use (ρ, θ, ϕ).
- When done, give your exam to your proctor, who will mark your name off on a photo roster.

Completely fill in exactly one of the bubbles for your multiple choice answers for Problem 1 Parts (1) to (10).

(1) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(2) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(3) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(4) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(5) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(6) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(7) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(8) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(9) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

(10) Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ

For Grader Use Only | **/30**

Math 2400, Final Exam
December 18, 2019

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- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like $\ln(3)/2$ as is.
- For cylindrical coordinates use (r, θ, z), and for spherical coordinates use (ρ, θ, ϕ).
- When done, give your exam to your proctor, who will mark your name off on a photo roster.

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ____________________

For Grader Use Only

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	30	7	7	10	8	5	3	10	10	10	100
Score:											

1. Multiple Choice: For the following multiple choice questions, no partial credit is given. **Fill in your answer on the bubble sheet.**

(1) (3 points) **Fill in your answer on the bubble sheet.**
Suppose $f(x, y, z) = x + y^2 + z^2$, and let S be the level surface $f(x, y, z) = 8$. Find the equation of the tangent plane to S at the point $(-2, 1, 3)$.

(A) $(x + 2) + 2(y - 1) + 6(z - 3) = 0$

(B) $(x + 2) + 2(y - 1) + 2(z - 3) = 0$

(C) $(x - 2) + 2(y + 1) + 6(z + 3) = 0$

(D) $(x - 2) + 2(y + 1) + 6(z + 3) = 8$

(E) $(x - 2) + 2(y + 1) + 6(z + 8) = 0$

(F) $(x + 2) + 2(y - 1) + 6(z - 3) = 8$

(2) (3 points) **Fill in your answer on the bubble sheet.**
Find the parametrization of the part of the elliptic paraboloid $y = 4x^2 + z^2 - 4$ that lies inside the cylinder $x^2 + z^2 = 4$.

(A) $\langle x,\ 4x^2 + z^2 - 4,\ z\rangle$ for $-1 \leq x \leq 1$ and $-2 \leq z \leq 2$

(B) $\langle x,\ x^2 + z^2,\ z\rangle$ for $-2 \leq x \leq 2$ and $0 \leq z \leq 4$

(C) $\langle x,\ 4 - x^2 - z^2,\ z\rangle$ for $-2 \leq x \leq 2$ and $-\sqrt{4 - x^2} \leq z \leq \sqrt{4 - x^2}$

(D) $\langle r\cos\theta,\ r^2 + 3r^2\cos^2\theta - 4,\ r\sin\theta\rangle$ for $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$

(E) $\langle r\cos\theta,\ r^2 - 4,\ 2r\sin\theta\rangle$ for $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$

(F) $\left\langle \frac{1}{2}r\cos\theta,\ r^2 - 4,\ r\sin\theta\right\rangle$ for $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$

(3) (3 points) **Fill in your answer on the bubble sheet.**

Suppose

$$f(x, y) = ye^{-x} + 3x.$$

Find the direction of the maximum rate of increase of $f(x, y)$ at $(0, 1)$.

(A) $\langle 2, 1\rangle$

(B) $\langle -2, -1\rangle$

(C) $\langle 3, 0\rangle$

(D) $\langle -3, 0\rangle$

(E) $\langle 2e^{-1}, e\rangle$

(F) $\langle -2e^{-1}, -e\rangle$

(4) (3 points) **Fill in your answer on the bubble sheet.**

Find the following limit, if it exists.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 2y^2}{x^2 + y^2}$$

(A) 0

(B) 1

(C) -1

(D) 2

(E) -2

(F) The limit does not exist.

(5) (3 points) **Fill in your answer on the bubble sheet.**

Let

$$f(x,y) = \begin{cases} \dfrac{x+2}{x^2+y^2+1}, & \text{if } (x,y) \neq (0,0) \\ a & \text{if } (x,y) = (0,0) \end{cases}$$

Find a, such that the function $f(x,y)$ is continuous at $(0,0)$.

(A) 0

(B) 1

(C) -1

(D) 2

(E) -2

(F) There is no a for which f is continuous at (0,0).

(6) (3 points) **Fill in your answer on the bubble sheet.**

Let

$$f(x,y) = (x^3 - x)(y^2 - 1).$$

Find $f_{xy}(x,y)$.

(A) $(x^3 - x)(2y)$

(B) $(3x^2 - 1)(2y)$

(C) $(3x^2 - 1)(y^2 - 1) + (x^3 - x)(2y)$

(D) $(3x^2 - 1)(y^2 - 1)$

(E) 0

(F) $6x^2y + 2y - 3x^2 - 1$

(7) (3 points) **Fill in your answer on the bubble sheet.**

Let S be the surface parametrized by $\vec{r}(\theta, z) = \langle 3\cos(\theta), 3\sin(\theta), z\rangle$, for $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 2$. Evaluate

$$\iint_S 1\, dS.$$

(A) π

(B) 2π

(C) 3π

(D) 9π

(E) 12π

(F) 18π

(8) (3 points) **Fill in your answer on the bubble sheet.**

Let

$$\vec{F}(x, y, z) = \langle xyz, \quad xy + yz + zx, \quad x + y + zy\rangle.$$

Find curl$\vec{F}$.

(A) $\langle yz, \quad x + z, \quad 1\rangle$

(B) $\langle 1 - x - y + z, \quad -1 + xy, \quad y + z - xz\rangle$

(C) $\langle 1 + x + y, \quad xy, \quad xz\rangle$

(D) $\langle yz + y + z + 1, \quad xz + x + z + 1, \quad xy + x + y + 1\rangle$

(E) $\langle 1, \quad 1, \quad 1\rangle$

(F) $\langle y + z, \quad x + z, \quad x + y\rangle$

(9) (3 points) **Fill in your answer on the bubble sheet.**

Let

$$\vec{F}(x,y,z) = \left\langle x^3 + y^2,\ ze^{-y},\ x^2\sin(z)\right\rangle .$$

Find div$\vec{F}$

(A) $\left\langle 3x^2, -e^{-y}z, x^2\cos(z)\right\rangle$

(B) $\left\langle -e^{-y},\ -2x\sin(z),\ -2y\right\rangle$

(C) $\left\langle 3x^2+2y,\ -e^{-y},\ 2x\cos(z)\right\rangle$

(D) $3x^2 - ze^{-y} + x^2\cos(z)$

(E) $3x^2 + 2y - e^{-y} + 2x\cos(z)$

(F) $3x^2 + ze^{-y} - x^2\cos(z)$

(10) (3 points) **Fill in your answer on the bubble sheet.**

Let f be a scalar-valued function of three variables and $\vec{F}$ a vector field on $\mathbb{R}^3$. Which of the following must be true for all such f and $\vec{F}$? (Assume all functions and their components are polynomials.)

(A) $\operatorname{div}(\operatorname{div}\ f) = 0$

(B) $\operatorname{div}(\operatorname{grad}\ f) = 0$

(C) $\operatorname{curl}(\operatorname{div}\ f) = 0$

(D) $\operatorname{div}(\operatorname{curl}(\operatorname{curl}\ \vec{F})) = 0$

(E) $\operatorname{curl}(\operatorname{curl}(\operatorname{div}\ \vec{F})) = 0$

(F) $\operatorname{grad}(\operatorname{curl}\ \vec{F}) = 0$

2. (7 points) Convert the following integral from rectangular coordinates to cylindrical coordinates. Fill in all **7** blanks.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{18-2x^2-2y^2} yz\ dz\ dy\ dx$$

$$= \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{}\ dz\ dr\ d\theta$$

3. (7 points) Convert the following integral from spherical coordinates to rectangular coordinates. Fill in all **7** blanks.

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{2\sqrt{2}} \sin\phi\ \ d\rho\, d\phi\, d\theta$$

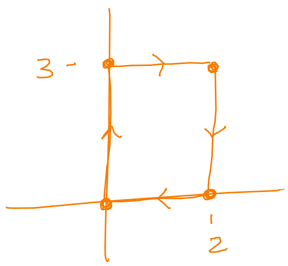
$$= \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{}\ dz\ dy\ dx$$

4. (10 points) Evaluate the integral

$$\int_C \left(xy^2 + y\right)\, dx + \left(2x^2y + e^{y^2}\right)\, dy$$

where C is boundary of the rectangle in the xy-plane oriented **clockwise** with vertices $(0,0)$, $(0,3)$, $(2,3)$, and $(2,0)$.

Draw C



$$\boxed{\int \underline{\nabla f} \cdot d\vec{r}} = f \Big|_{r(a)}^{r(b)}$$

$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \boxed{\int_{\underline{\partial S}} \vec{F} \cdot d\vec{r}}$$

$$\hookrightarrow \iint_S Q_x - P_y \, dA = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\iiint_E \text{div}\, \vec{F}\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$

Is $\vec{F} = \nabla f$? No because curl $\vec{F} \neq \vec{0}$. can't use FTLI

Stokes' / Green's

5. (8 points) For the following function, find all local maximums, local minimums, and saddle points.

$$f(x, y) = x^4 - 2x^2 + y^3 - 3y$$

6. (5 points) Consider the vector field $\vec{F}$ on $\mathbb{R}^2$ given by

$$\vec{F}(x,y) = \langle \pi\cos(\pi x) + y,\, x + 2y\rangle\,.$$

Find a potential function $f(x,y)$ for $\vec{F}(x,y)$ such that $\nabla f = \vec{F}$.

$\int \nabla f \cdot d\vec{r} = f\Big|_{r(a)}^{r(b)}$

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}$

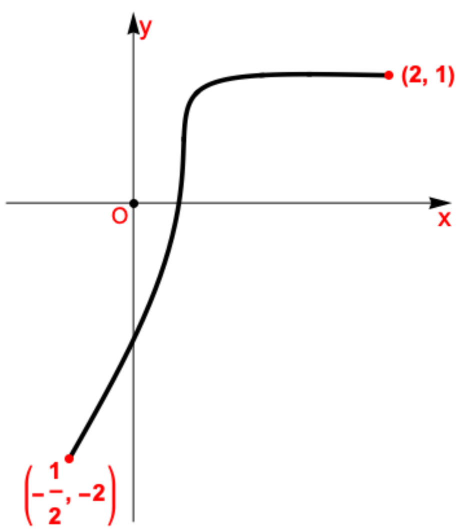
$\hookrightarrow \iint_S Q_x - P_y\, dA = \int_{\partial S} \vec{F}\cdot d\vec{r}$

$\iiint_E \text{div}\,\vec{F}\; dV = \iint_{\partial E} \vec{F}\cdot d\vec{S}$

7. (3 points) Let $\vec{F} = \nabla g$ where $g(x,y) = e^{\cos(\pi x)} + xy$. Evaluate the integral

$$\int_C \vec{F}\cdot d\vec{r},$$

where C is the path pictured below from $\left(-\frac{1}{2}, -2\right)$ to $(2,1)$.



FTLI because

$\vec{F} = \nabla g$

8. (10 points) Let S be the helicoid parameterized by

$$\vec{r}(u,v) = \langle u \sin v,\ 2v,\ u\cos v \rangle \quad \text{for } 0 \le u \le 1, \quad 0 \le v \le \pi,$$

oriented in the direction of the positive y-axis. Let $\vec{F}$ be a vector field given by

$$\vec{F} = xy\vec{i} + (y^2+1)\vec{j} + yz\vec{k}.$$

Evaluate $\iint_S \vec{F} \cdot d\vec{S}$.

$\operatorname{div} \vec{F} = y + 2y + y \neq 0$

$\vec{F}$ not the curl of another vector field. Not Stokes'

S is not closed, ie not the boundary of a solid E. Not Divergence Theorem

$$\int \nabla f \cdot d\vec{r} = f \Big|_{r(a)}^{r(b)}$$

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\hookrightarrow \iint_S Q_x - P_y \, dA = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\iiint_E \operatorname{div} \vec{F} \, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$

Direct computation

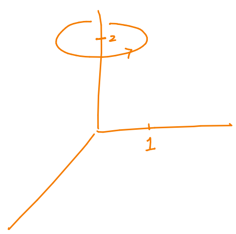
$$\iint_{u,v} \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dA$$

9. (10 points) Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ given by

$$\vec{F} = (\cos x + y)\vec{i} + (e^y + xz^2)\vec{j} + (2z^2 + yx)\vec{k}.$$

Let C be a circle of radius 1 centered at $(0,0,2)$ lying on the plane $z = 2$, which is oriented **counterclockwise** when viewed from above. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

Draw C



$$\boxed{\int \nabla f \cdot d\vec{r}} = f\Big|_{r(a)}^{r(b)}$$

$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \boxed{\int_{\partial S} \vec{F} \cdot d\vec{r}}$$

$$\hookrightarrow \iint_S Q_x - P_y \, dA = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\iiint_E \text{div}\, \vec{F} \, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos(x+y) & e^y + xz^2 & 2z^2 + xy \end{vmatrix}$$

$$= \langle x - x, -(y - 0), z^2 + \sin(x+y) \rangle$$

$$= \langle 0, -y, z^2 + \sin(x+y) \rangle$$

$\text{curl}\, \vec{F} \neq \vec{0}$ so $\vec{F}$ not conservative

Not Fundamental Thm of Line Integrals

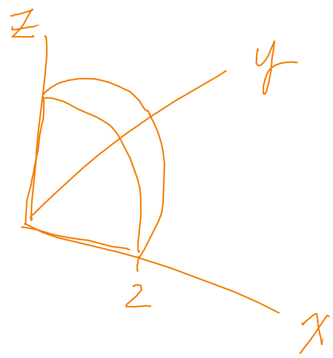
But C is a closed curve, ie the boundary of the disk inside, so use Stokes' with S: $\vec{r}(u,v) = \langle u, v, 2 \rangle$, $u^2+v^2 \le 1$ oriented upward.

10. (10 points) Let

$$\vec{F}(x,y,z) = \left(x^3 + e^{y^2+z^2}\right)\vec{i} + \left(\cos(x^4) + y^3\right)\vec{j} + \left(\ln(x^2+4) + z^3\right)\vec{k}$$

be a vector field on $\mathbb{R}^3$, region E be the part of the solid sphere $x^2+y^2+z^2 \le 4$ in the **first octant**, and S be the boundary of E oriented outward. Find the total flux of $\vec{F}$ through S:

$$\iint_S \vec{F} \cdot d\vec{S}.$$



$$\int \nabla f \cdot d\vec{r} = f \Big|_{r(a)}^{r(b)}$$

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\hookrightarrow \iint_S Q_x - P_y \, dA = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\iiint_E \text{div } \vec{F} \, dV = \boxed{\iint_{\partial E} \vec{F} \cdot d\vec{S}}$$

$\text{div } \vec{F} = 3x^2 + 3y^2 + 3z^2 \neq 0$

no Stokes' b/c $\vec{F}$ not incompressible

But S is the closed boundary of the solid E

So use Divergence thm

$$\iiint 3(x^2+y^2+z^2)\, dV$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Thank You

- **For hybrid sections, only:** MWF are in person until Fall Break, and TTh are remote for the duration of the semester. M-F are remote after Fall Break.
- **For remote sections, only:** M-F are remote for the duration of the semester.
- WebAssign assignments will be due each Sunday for the previous week's topics. However, you should complete each assignment during the week after the topic has been covered.
- The textbook section coverage for each biweekly quiz can be found at the end of this document.

Monday	Tuesday	Wednesday	Thursday	Friday
Aug 24th 9.1 - 3D Coordinate Systems	25th 9.2 - Vectors Check-in 1 A: Intro to Mathematica (at home) A: Guidelines for 3D Graphing (at home)	26th 9.2 (cont.) A: Vectors and Mathematica (at home)	27th P: Mathematica and 3D Graphing HW 1 due	28th 9.3 - Dot Product Check-in 2 Check-in 3 (Proctorio check, due Sunday)
31st 9.4 - Cross Product	Sep 1st 9.5 - Equations of Lines and Planes QUIZ 1	2nd 9.6 - Functions and Surfaces	3rd P: What is This Thing? #1 HW 2 due	4th 9.7 - Cylindrical and Spherical Coordinates Check-in 4
7th Labor Day No Class	8th A: Quadric Surfaces Check-in 5	9th 10.1 - Vector Functions and Space Curves	10th P: Parametrized Curves and Surfaces HW 3 due	11th 10.2 - Derivatives and Integrals of Vector Functions Check-in 6
14th 10.3 - Arc Length	15th 10.5 - Parametric Surfaces A: Parametric Matching QUIZ 2	16th 10.5 (cont.)	17th P: Introduction to Line Integrals HW 4 due	18th 11.1 - Functions of Several Variables Check-in 7
21st 11.2 - Limits and Continuity	22nd 11.2 (cont.) Check-in 8	23rd 11.3 - Partial Derivatives	24th P: Limits and Polar Coordinates HW 5 due	25th 11.4 - Tangent Planes and Linear Approximations Check-in 9

Monday	Tuesday	Wednesday	Thursday	Friday
28th 11.5 - The Chain Rule	29th A: What is the Derivative of This Thing? A: Composition of Functions QUIZ 3	30th 11.6 - Directional Derivatives and the Gradient Vector	Oct 1st P: Gradient Graphically HW 6 due	2nd 11.6 (cont.) Check-in 10
5th 11.7 - Maximum and Minimum Values	6th 11.7 (cont.) Check-in 11	7th 11.8 - Lagrange Multipliers	8th P: Optimization HW 7 due	9th 12.1 - Double integrals over rectangles 12.2 - Iterated Integrals Check-in 12
12th 12.2 (cont.) 12.3 - Double Integrals Over General Regions	13th 12.3 (cont.) QUIZ 4	14th 12.4 - Double Integrals in Polar Coordinates	15th P: Slices vs. Skyscrapers and Order of Integration HW 8 due	16th 12.4 (cont.) Check-in 13
19th 12.5 - Applications of Double Integrals	20th 12.6 - Surface Area Check-in 14	21st 12.7 - Triple Integrals	22nd P: Applications of Multiple Integrals HW 9 due	23rd 12.7 (cont.) Check-in 15
26th 12.8 - Triple Integrals in Cylindrical and Spherical Coordinates	27th 12.8 (cont.) QUIZ 5	28th 12.8 (cont.)	29th P: Introduction to Surface Integrals HW 10 due	30th 12.9 - Change of Variables in Multiple Integrals Check-in 16
Nov 2nd 12.9 (cont.)	3rd 13.1 - Vector Fields A: Vector Field Matching Check-in 17	4th 13.2 - Line Integrals Over *Scalar* Functions	5th P: Line Integrals Over Vector Fields HW 11 due	6th 13.2 - Line Integrals Over *Vector* Fields Check-in 18
9th 13.3 - Fundamental Theorem of Calculus for Line Integrals	10th 13.3 (cont.) QUIZ 6	11th 13.4 - Green's Theorem	12th P: What is This Thing? #2 (Line Integrals) HW 12 due	13th 13.5 - Curl and Divergence Check-in 19

Monday	Tuesday	Wednesday	Thursday	Friday
16th 13.5 (cont.)	17th A: Conservative or Not? Check-in 20	18th 13.6 - Surface Integrals Over *Scalar* Fields	19th P: Introduction to Flux HW 13 due	20th 13.6 - Surface Integrals Over *Vector* Fields Check-in 21 (last)
23rd 13.7 - Stokes' Theorem	24th Review QUIZ 7 QUIZ 8	25th Remote office hours during class time	26th Fall Break No Class	27th Fall Break No Class
30th 13.7 (cont.) Corrections for Quizzes 7+8 due	$\boxed{\text{Dec 1st}}$ 13.8 - Divergence Theorem	2nd 13.8 (cont.)	3rd P: What is This Thing? #3 (Types of Integrals) HW 14 due	4th A: Fundamental Theorem Practice A: Fundamental Theorem Matching
7th Review	8th Fall Reading Day No Class	9th	10th	11th

Quiz Coverage

All quizzes are 30 minutes long. Quizzes 1-8 will be open between 7pm and 10pm on the date indicated below. Quizzes 9 and 10 will be open during the final exam time on the University's schedule.

- Quiz 1 (Sep 1): 9.1-9.3
- Quiz 2 (Sep 15): 9.4-10.2
- Quiz 3 (Sep 29): 10.3-11.4
- Quiz 4 (Oct 13): 11.5-12.1
- Quiz 5 (Oct 27): 12.2-12.7
- Quiz 6 (Nov 10): 12.8-13.2
- Quiz 7 (Nov 24): 13.3-13.6
- Quiz 8 (Nov 24): comprehensive
- Quiz 9 (during the assigned final exam time): 13.7-13.8
- Quiz 10 (during the assigned final exam time): comprehensive