Previous projects: Prior to doing this project, it is helpful (but not essential) if students have done the quadric surfaces matching project.

Background content: Prior to doing this project, students should have a working knowledge of the following:

- Calculating the gradient vector of a potential function
- Recognizing the graph of the level curves of a scalar function
- Identifying the graph of a scalar function given its formula
- Know factually that at each point the gradient vector points perpendicular to the level curve of its potential function.
- Know factually that at each point the gradient vector points in the direction of fastest increase of the potential function.

Philosophy behind this project:
One purpose of the project is to give a visual and tactile experience of the concepts that the gradient vector ∇f points perpendicular to the level curves of f and points in the direction of fastest increase of f. We present this project before students have officially seen graphs of vector fields, so the project also introduces these types of graphs and how to create these graphs from the formula for a vector field. Presentation of graphs of vector fields at this point allows students to understand the gradient as more than just a vector at each point, but also as a function that can be visualized globally.

Learning Goals: Internalize the connections between five different perspectives of a potential function. These five perspectives are the graph of the potential function, the formula for the potential function, the graph of the level curves of the potential function, the formula for the gradient vector field of the potential function, and the graph of the gradient vector field of the potential function.

Implementation Notes: The project is designed to be printed and cut up into cards. There are three sets of five different types of cards (graphs of potential functions, formulas for potential functions, graphs of the level curves of potential functions, formulas for the gradient vector field of the potential functions, graphs of the gradient vector field of the potential functions). The goal is to match them into three sets of five card, each set representing the same potential function. Ideally the graphs of the vector fields will be printed on acetate. This allows students to lay the graphs of the vector fields on top of the graphs of the level curves to see if the gradient vectors point perpendicular to the level curves at each point.

It is possible in this project to match the cards into sets intuitively (figure out which graphs feel like they go together) Also, if students match $\heartsuit$ to A and A to 2, they can escape seeing the direct connection between $\heartsuit$ and 2. Therefore, the instructor must ask questions that require the students to articulate what precise features of the graphs and formulas they are using to verify that the match is correct. Suggested questions are given below.

1. Sample questions for matching graphs of the potential function ($\heartsuit, \diamondsuit, \clubsuit$) to the level curves ($A$, B, C) (Note that this is a review, since they have seen this type of question in a previous chapter):

- How do construct the graphs of the level curves if you start with a graph of the potential function? (I slice the potential function perpendicular to the z-axis and that cross-section gets graphed in the xy-plane as a level curve)
- The slices of $\heartsuit$ and $\diamondsuit$ perpendicular to the z-axis are both circular. How do I figure out which goes with which? (Since the graph of $\diamondsuit$ climbs increasingly sharply, the level curves get closer and closer together. On the other hand, the sides of the cone go up linearly, so the spacing between the circles is constant)
- On C, does the dark orange correspond to higher or lower levels than the lighter-orange? (The responses "I can see on $\clubsuit$ that the higher levels are darker orange" is inadequate, since you cannot guarantee that the person who created the graphs was consistent with the colors. Instead, the answer has to be "Since the heights along the x-axis are higher on the graph of the potential function than along the y-axis, the curves crossing the x-axis in the graph of the level curves must be higher, and those are dark orange".)
- If you look at the level curves, can you see it as a bird's-eye view of the potential function, and visualize the surface in 3D, popping out from and receding behind the paper?

2. Sample questions for matching gradient vector fields $(1, 2, 3)$ to level curves (A, B, C):

   - The gradient vector fields are printed on acetate so they are transparent. Why do you think we did that? (So I can lay it on top of the level curves and see and feel that the gradient vector at every point is perpendicular to its level curve.)
   - The identifying numbers $(1, 2, 3)$ on the gradient fields appear at each corner, so you have to rotate the field to get it to point in the correct direction. Because of radial symmetry, changing the orientation doesn't matter for 1 and 2, but how do I figure out which way to orient the gradient vector field in 3? (First recall that I figured out previously that the dark orange is the high point. Now recall that the gradient vector points in the direction of fastest *increase*. This tells me that I must orient the vector field so the arrows point from lighter orange to darker orange.)

3. Sample questions for matching graphs of potential functions $(\heartsuit, \diamondsuit, \clubsuit)$ to graphs of the gradient vector field $(1, 2, 3)$:

   - Gradient fields 1 and 2 have vectors pointing directly away from the origin. What does that say about the graph of the potential function? (That it has radial symmetry, and that to climb the fastest, you should move directly away from the origin.)
   - Graphs 1 and 2 are kind of similar, but in graph 1 the arrows get longer as you move away from the origin. What does this mean about the graph of the potential function? (The length of the gradient vector gives information about the rate of increase of the potential function. So for graph 1, the potential function climbs more quickly as you move away from the origin)
   - If you look at the gradient vector field, can you see it as a bird's-eye view of the potential function, and visualize the surface in 3D, popping out from and receding behind the paper? Where are the flat areas? Try this in particular on graph 3, forcing you to pay attention to which axes have the peaks and which have the valleys, to rotate the gradient vector field to the proper orientation.

- I've heard students say that since the gradient vector points perpendicular to the potential function, it should be a 3D vector, not a 2D vector. Are these students correct? How do you respond to them? (The gradient vector points perpendicular to *level curves*, not the potential function. It points in the direction of fastest increase of the potential function, but that just tells you which direction to go on the bird's-eye view map, it doesn't actually point up or down. Of course there is a way to find the vector perpendicular to the surface of the potential function itself. That vector is the 3D vector $\langle f_x, f_y, -1\rangle$, which is different from the 2D gradient vector $\langle f_x, f_y\rangle$.

4. Sample questions for matching formulas of potential functions (a, b, c) to graphs of potential functions: $(\heartsuit, \diamondsuit, \clubsuit)$ (This is something of a review, since students practiced a similar skill in the quadric surfaces project):

   - Looking at the three potential function formulas a, b and c, which ones have radial symmetry and why?
   - What was the main technique we used for matching formulas to graphs in the quadric surfaces project? (Traces.)
   - What are the xy, xz and yz traces for each of the potential functions? Find them using the formulas, and verify that they match the graphs.
   - Which traces are not useful for distinguishing A from B?

5. Sample questions for matching graphs of level curves (A, B, C) to formulas for the potential functions (a, b, c):

   - The level curves are traces of the potential function. Which traces are they and how do I find them? (They are traces perpendicular to the xy plane, and represent constant values of z.)
   - How do I relate the potential function to the level curves? (plug in constant z values and see what shape you get.)
   - Find the level curves for each of the three potential functions using the formula for the potential function.

6. Sample questions for matching formulas for the gradient vector fields (α, β, γ) to the graphs of the vector fields $(1, 2, 3)$ (Note that they will practice this skill in an upcoming project, vector field matching. Don't expect them to be sophisticated at this yet since it is their first significant content with graphs of vector fields):

   - One strategy for matching the vector fields to the formulas is to select a point, substitute into the formulas, and see which graph it matches with.
   - Another strategy is to find the magnitude of the vector field from the formula, and see if it is constant or increasing/decreasing in length as we move away from the origin.

7. Sample questions for matching formulas of gradient vector fields (α, β, γ) to formulas for potential functions a, b, c: How do I find a gradient vector field from a potential function? (Take the partial derivatives, and form the vector $\langle f_x, f_y\rangle$).